Exam 2 Review Problems

Exam 2 is Tuesday 4/4 and will cover chapters 4-10 in Book of Proof. The following questions are meant to provide an additional opportunity to practice this material.

Prove the following statements. Use complete sentences.

1. Suppose a, b, c, d are positive integers. If $a \mid b$ and $c \mid d$ then $ac \mid bd$.

Solution: (Direct proof) Let us assume that $a \mid b$ and $c \mid d$. Then $b = am$ and $d = cn$ for some $m, n \in \mathbb{Z}$. Now,

$$bd = (am)(cn) = (mn)(ac).$$

Since $mn \in \mathbb{Z}$, it follows that $ac \mid bd$. □

2. Suppose $a, b, c \in \mathbb{Z}$, and $n \in \mathbb{N}$. If $a \equiv b \pmod{n}$, and $a \equiv c \pmod{n}$, then $2a \equiv b + c \pmod{n}$.

Solution: (Direct proof) Let us assume that $a \equiv b \pmod{n}$, and $a \equiv c \pmod{n}$. By definition, $n \mid (a - b)$ and $n \mid (a - c)$. That is, $a - b = nx$ and $a - c = ny$ for some $x, y \in \mathbb{Z}$. Now,

$$2a - (b + c) = (a - b) + (a - c) = nx + ny = n(x + y).$$

Since $x + y \in \mathbb{Z}$, this shows that $n \mid 2a - (b + c)$. Therefore, by definition, $2a \equiv b + c \pmod{n}$. □

3. Suppose A and B are sets. Then $A - (A - B) = A \cap B$.

Solution: We prove the statement using only definitions and logical equivalences.

$$\begin{aligned}
A - (A - B) &= \{x \ : \ (x \in A) \wedge \sim (x \in (A - B))\} \\
&= \{x \ : \ (x \in A) \wedge \sim ((x \in A) \wedge \sim (x \in B))\} \\
&= \{x \ : \ (x \in A) \wedge (\sim (x \in A) \vee (x \in B))\} \\
&= \{x \ : \ ((x \in A) \wedge \sim (x \in A)) \vee ((x \in A) \wedge (x \in B))\} \\
&= \{x \ : \ (x \in A) \wedge \sim (x \in A)\} \cup \{x \ : \ (x \in A) \wedge (x \in B)\} \\
&= \emptyset \cup (A \cap B) \\
&= A \cap B
\end{aligned}$$

□

4. The number $\log_2 3$ is irrational.
Hint: Use proof by contradiction and the fact that $\log_2 3 > 0$.

Solution: (Contradiction) Assume for the sake of contradiction that $\log_2 3$ is rational. Then $\log_2 3 = a/b$ for some $a, b \in \mathbb{Z}$. Moreover, since $\log_2 3 > 0$, we may assume a and b are both positive integers. Since $\log_2 3 = a/b$, it follows that $2^{a/b} = 3$. Raising both sides of this equation to the power b, we see

$$2^a = 3^b.$$

Since a and b are both positive integers, 2^a is the product of even integers and is even, while 3^b is the product of odd integers and is odd $\Rightarrow\Leftarrow$. This is a contrdiction – an even integer cannot equal an odd integer. Therefore our assumption that $\log_2 3$ is rational must be false. This proves that $\log_2 3$ is irrational. $\square$

5. The number $\sqrt{6}$ is irrational.

Solution: (Contradiction) Assume for the sake of contradiction that $\sqrt{6}$ is rational. Then $\sqrt{6} = a/b$ for some $a, b \in \mathbb{Z}$. Without loss of generality, assume that the fraction a/b is reduced and, in particular, that a and b are not both even.

Now since $\sqrt{6} = a/b$, it follows that $\sqrt{6}b = a$ and, squaring both sides, we have

$$a^2 = 6b^2 = 2(3b^2). \tag{1}$$

Since $3b^2 \in \mathbb{Z}$, it follows that a^2 is even and, since the product of two odd integers is odd, a must be even. Set $a = 2n$ for some $n \in \mathbb{Z}$. Now equation (1) says

$$(2n)^2 = 2(3b^2),$$
$$2n^2 = 3b^2.$$

Since $n^2 \in \mathbb{Z}$, it follows that $3b^2$ is even and, since 3 is odd this implies b must be even $\Rightarrow\Leftarrow$. This contradicts the fact that a and b are not both even. Thus our assumption that $\sqrt{6}$ is rational must be false. This proves that $\sqrt{6}$ is irrational. $\square$

6. There exists a set X such that $X \cap \mathcal{P}(X)$ is not empty.
Hint: What element(s) is in $\mathcal{P}(X)$ no matter what X is?

Solution: (Existence) Let X be any set that contains the empty set as an element, say $X = \{\emptyset\}$. Then $\mathcal{P}(X) = \{\emptyset, \{\emptyset\}\}$ and

$$X \cap \mathcal{P}(X) = \{\emptyset\} \cap \{\emptyset, \{\emptyset\}\} = \{\emptyset\}$$

is not empty. It contains thee empty set as an element. $\square$.

7. Suppose n is an integer. If $3 \nmid n$, then $3 \mid (n^2 - 1)$.
Hint: Divide into cases.

Solution: (Direct proof by cases) Suppose $3 \nmid n$. Then $3 \nmid (n - 0)$ and so, by definition, $n \not\equiv 0 \pmod 3$. Thus, either $n \equiv 1 \pmod n$ or $n \equiv 2 \pmod n$.

In the case that $n \equiv 1 \pmod 3$, we have $n - 1 = 3k$ for some $k \in \mathbb{Z}$. Adding 2 to both sides gives $n + 1 = 3k + 2$. Therefore

$$n^2 - 1 = (n + 1)(n - 1) = (3k + 2)(3k) = 3[k(3k + 2)].$$

Since $k(3k + 2)$ is an integer, this shows that $3 \mid n^2 - 1$.

In the case that $n \equiv 2 \pmod 3$, we have $n - 2 = 3k$ for some $k \in \mathbb{Z}$. Adding 2 to both sides gives $n = 3k + 2$. Therefore

$$n^2 - 1 = (3k + 2)^2 - 1 = (9k^2 + 12k + 4) - 1 = 3(3k^2 + 4k + 1).$$

Since $3k^2 + 4k + 1$ is an integer, this shows that $3 \mid n^2 - 1$.
In either case, $3 \mid n^2 - 1$. This completes the proof. $\square$.

8. For all integers $n \geq 1$,

$$3 + 3^2 + 3^3 + \ldots + 3^n = \sum_{i=1}^{n} 3^i = \frac{3^{n+1} - 3}{2}.$$

Solution: (Mathematical induction) We proceed by inducton. First, observe that the equation is true for $n = 1$.

$$3 = \frac{3^{1+1} - 3}{2} = \frac{9 - 3}{2} \quad \checkmark$$

Now let's assume that

$$\sum_{i=1}^{n} 3^i = \frac{3^{n+1} - 3}{2}$$

and use this to show that

$$\sum_{i=1}^{n+1} 3^i = \frac{3^{(n+1)+1} - 3}{2}.$$

We have

$$\begin{aligned}
\sum_{i=1}^{n+1} 3^i &= \sum_{i=1}^{n} 3^i + 3^{n+1} \\
&= \frac{3^{n+1} - 3}{2} + 3^{n+1} \\
&= \frac{3 \cdot 3^{n+1} - 3}{2} \\
&= \frac{3^{(n+1)+1} - 3}{2}.
\end{aligned}$$

Thus, by mathematical induction, the equation is true for all $n \in \mathbb{N}$. $\square$

9. Suppose $x, y \in \mathbb{R}$. If

$$xy - x^2 + x^3 \geq x^2y^3 + 4,$$

then $x \geq 0$ or $y \leq 0$.
Hint: Try proving the contrapositive statement.

Solution: (Contrapositive) We will prove the contrapositive statement:

$$\text{If } x < 0 \text{ and } y > 0, \text{ then } xy - x^2 + x^3 < x^2y^3 + 4.$$

Assume $x < 0$ and $y > 0$. Since the product of an odd number of negative real numbers is negative and the product of an even number of negative real numbers is positive, we have

$$xy < 0, \quad -x^2 < 0, \quad x^3 < 0 \tag{2}$$

and

$$0 < x^2y^3, \quad 0 < 4. \tag{3}$$

Now sum all of the inequalities in (2) and (3) to see

$$xy - x^2 + x^3 < x^2y^3 + 4.$$

□