§1.1 Introduction to Sets

Def: A **set** is a collection of objects called **elements**.

Notation: $\in$ = "is in"

e.g. $B = \{ 4, -2, \underbrace{\{7, 8, 9\}}_{\text{set with 3 elements}}, 0 \}$ — set with 4 elements

$4 \in B$ $\quad$ $1 \notin B$

$\{7,8,9\} \in B$ $\quad$ $8 \notin B$

$\{9,8,7\} \in B$ $\quad$ $8 \in \{7,8,9\}$

Order doesn't matter.
Two sets are **equal** if they contain the same elements.

Notation: $|B| = 4$

Cardinality or size
= # of elements

Infinite sets: Natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$

Integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

Empty set $\{\} = \emptyset$, $|\emptyset| = 0$

ex. $|\{\emptyset\}| = 1$ $\qquad$ $|\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}| = 3$

$|\{\emptyset, \underbrace{\{\emptyset\}}\}| = 2$ $\qquad$ $\left(|\{\{\}, \{\{\}\}, \{\{\{\}\}\}\}| = 3 \right)$

A folder that contains an empty folder

Set Builder Notation

$:$ & $|$ interchangeable

Expression : Rule

$\{1, 3, 5, 7, \dots\} = \{2n-1 : n \in \mathbb{N}\} = \{2n-1 \mid n \in \mathbb{N}\}$

"The set of all things of the form $2n-1$ such that $n \in \mathbb{N}$"

or $\{n : n \text{ is an odd positive integer}\}$

$\{n \in \mathbb{N} : n \text{ is odd}\}$

ex. List elements: $\{n^2 : n \in \mathbb{Z}\} = \{1, 4, 9, 16, \dots\}$

$\{y \in \mathbb{Z} : 1 < y \le 5\}$

"Solve $\cot x = 0$" ?

(**ex.** $\{x \in \mathbb{R} : \cot x = 0\} = \{\frac{\pi}{2} + n\pi : n \in \mathbb{N}\}$)

$\{x \in \mathbb{R} : x^2 = 2\} = \{\sqrt{2}, -\sqrt{2}\}$

$\{x \in \mathbb{Z} : x^2 = 3\} = \emptyset$

$\{(a,b,c) : a,b,c \in \mathbb{N} \text{ and } a^2+b^2=c^2\}$

(Fermat's Last Thm) $\{(a,b,c) : a,b,c \in \mathbb{N} \text{ and } a^3+b^3=c^3\} = \emptyset$

$\{7a + 11b : a,b \in \mathbb{Z}\} = \mathbb{Z}$

Note: This set contains only integers and every integer is in this set.
(For any $n \in \mathbb{Z}$, $7(-3n) + 11(2n) = n$)

D. Sketch the following sets of points in the x-y plane.

39. $\{(x,y) : x \in [1,2], y \in [1,2]\}$

40. $\{(x,y) : x \in [0,1], y \in [1,2]\}$

41. $\{(x,y) : x \in [-1,1], y = 1\}$

42. $\{(x,y) : x = 2, y \in [0,1]\}$

43. $\{(x,y) : |x| = 2, y \in [0,1]\}$

44. $\{(x,x^2) : x \in \mathbb{R}\}$

45. $\{(x,y) : x,y \in \mathbb{R}, x^2+y^2 = 1\}$

46. $\{(x,y) : x,y \in \mathbb{R}, x^2+y^2 \le 1\}$

47. $\{(x,y) : x,y \in \mathbb{R}, y \ge x^2 - 1\}$

48. $\{(x,y) : x,y \in \mathbb{R}, x > 1\}$

49. $\{(x, x+y) : x \in \mathbb{R}, y \in \mathbb{Z}\}$

50. $\{(x, \frac{x^2}{y}) : x \in \mathbb{R}, y \in \mathbb{N}\}$

51. $\{(x,y) \in \mathbb{R}^2 : (y-x)(y+x) = 0\}$

52. $\{(x,y) \in \mathbb{R}^2 : (y-x^2)(y+x^2) = 0\}$

1.2 Cartesian Product

Definition 1.1 An **ordered pair** is a list (x,y) of two things x and y, enclosed in parentheses and separated by a comma.

Note: any list of 2 things inside parentheses is an ordered pair.

e.g. $((2,5),(5,2))$ is an ordered pair of ordered pairs.

Definition 1.2 The **Cartesian product** of two sets A and B is another set, denoted as $A \times B$ and defined as $A \times B = \{(a,b) : a \in A, b \in B\}$.

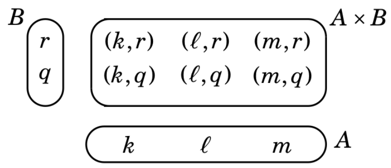

Figure 1.1. A diagram of a Cartesian product

Fact 1.1 If A and B are finite sets, then $|A \times B| = |A| \cdot |B|$.

ex. LIST ELEMENTS OF $\{A, B\} \times \{1, 2\}$

ex. SKETCH $\mathbb{R} \times \mathbb{Z}$, $\mathbb{N} \times \mathbb{R}$, $\mathbb{Z} \times \mathbb{N}$.

DEF. GENERALIZATION:

FOR ANY $n \in \mathbb{N}$, $n \geq 2$, AN ORDERED n-TUPLE IS A LIST OF n OBJECTS.

THE CARTESIAN PRODUCT OF n SETS

$$A_1 \times A_2 \times \dots \times A_n = \{ (a_1, a_2, \dots, a_n) : a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n \}$$

$$\text{or } \{ (a_1, a_2, \dots, a_n) : a_i \in A_i \text{ for } i = 1, 2, \dots, n \}$$

Def. GIVEN A SET A AND $n \in \mathbb{N}$, THE CARTESIAN POWER

$$A^n = A \times A \times \dots \times A \quad (n \text{ TIMES})$$

$$= \{ (a_1, a_2, \dots, a_n) : a_1, a_2, \dots, a_n \in A \}$$

e.g. $\mathbb{R}^2$, $\mathbb{R}^3$

ex. $A = \{a, b, c, \dots, z\}$. WHAT IS A^3? A^4?

WHAT IS $|A^3|$? $|A^4|$?

ex. SKETCH $[0,1]^3$.

ex. SKETCH $\{ (x,y) \in \mathbb{R}^2 : y = \ln x \}$

Note: Our definition of a set as a collection of elements is fine for almost all practical purposes. However, it is insufficient. To be a set, one must be able to determine whether any particular object is or is not an element of that set. It is impossible to be both. This connects to the fact that every mathematical statement is either true or false — even when it is hard to determine, it cannot be both.

As an example of something that seems like a set but actually is not, consider
S = THE SET OF ALL SETS THAT DO NOT CONTAIN THEMSELVES AS AN ELEMENT.
= {A : A is not in A)

Many things are in S. For example, the set of natural numbers, integers, real numbers, {a,b,c}, etc.
QUESTION: Is S in S?

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