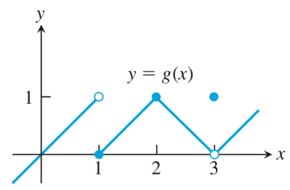
Final Exam

Answer all 12 questions for a total of 100 points. Calculators are not allowed. Write your solutions in the accompanying blue book and put a box around your final answers. Answers do not need to be simplified, and may include exponential and logarithmic expressions.

1. Use the graph $y = f(x)$ below to evaluate the following limits or explain why they do not exist.



(a) (2 points) $\lim_{x\to 1} g(x)$ DNE

(b) (2 points) $\lim_{x\to 2} g(x)$ 1

(c) (2 points) $\lim_{x\to 3} g(x)$ 0

2. Evaluate each of the following limits.

(a) (4 points) $\lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3} \cdot \frac{2+\sqrt{x^2-5}}{2+\sqrt{x^2-5}} = \lim_{x\to -3} \frac{2^2-(x^2-5)}{(x+3)(2+\sqrt{x^2-5})}$

$= \lim_{x\to -3} \frac{9-x^2}{(x+3)(2+\sqrt{x^2-5})} = \lim_{x\to -3} \frac{\cancel{(3+x)}(3-x)}{\cancel{(x+3)}(2+\sqrt{x^2-5})} = \frac{3}{2}$

(b) (4 points) $\lim_{x\to 0} \frac{\frac{1}{x-1}+\frac{1}{x+1}}{x} = \lim_{x\to 0} \frac{1}{x} \cdot \frac{x+1+x-1}{(x+1)(x-1)}$

$= \lim_{x\to 0} \frac{2\cancel{x}}{\cancel{x}(x+1)(x-1)} = -2$

(c) (4 points) $\lim_{x\to 1} \frac{e^{x-1}}{2+\ln x} = \frac{1}{2}$

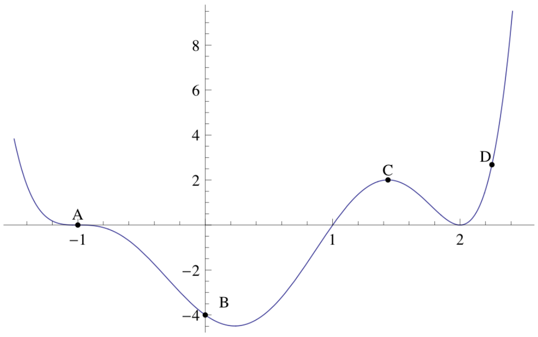
3. (6 points) Suppose $f(x) = 3x^2 + x$. Use the definition of the derivative (as a limit) to evaluate $f'(2)$.

$f'(2) = \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2} \frac{3x^2+x-14}{x-2} = \lim_{x\to 2} \frac{(3x+7)\cancel{(x-2)}}{\cancel{x-2}} = 13$

Alt: $f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{3(2+h)^2+(2+h)-14}{h}$

$= \lim_{h\to 0} \frac{\cancel{12}+12h+3h^2+\cancel{2}+h-\cancel{14}}{h} = \lim_{h\to 0} \frac{\cancel{h}(13+3h)}{\cancel{h}} = 13$

4. (12 points) Below is the graph of a function $y = f(x)$. Fill in a chart like the one below with $+$, $-$ or 0 to indicate whether f, f' and f'' are positive, negative or zero at each of the indicated points A, B, C and D.



	f	f'	f''
A	0	0	0
B	$-$	$-$	$+$
C	$+$	0	$-$
D	$+$	$+$	$+$

5. Compute the derivatives of the functions below. You do not need to simplify your answers.

(a) (4 points) $f(x) = \dfrac{e^x - e^{-x}}{x^2}$

$$\boxed{f'(x) = \frac{(e^x + e^{-x})x^2 - (e^x - e^{-x})2x}{(x^2)^2}}$$

(b) (4 points) $g(x) = \left(3x^2 - 2\sqrt{x} + \dfrac{1}{x}\right)^7$

$$\boxed{g'(x) = 7\left(3x^2 - 2\sqrt{x} + \frac{1}{x}\right)\left(6x - \frac{1}{\sqrt{x}} - \frac{1}{x^2}\right)}$$

(c) (4 points) $h(x) = x(2^x + \ln(4x^2))$

$$\boxed{h'(x) = 2^x + \ln(4x^2) + x\left(2^x \ln 2 + \frac{1}{4x^2} \cdot 8x\right)}$$

6. Evaluate the following indefinite integrals.

(a) (4 points) $\displaystyle\int \frac{(x+3)^2}{\sqrt{x}}\,dx$

$$= \int x^{-1/2}(x^2 + 6x + 9)\,dx = \int x^{3/2} + 6x^{1/2} + 9x^{-1/2}\,dx$$

$$= \boxed{\frac{2}{5}x^{5/2} + 4x^{3/2} + 18x^{1/2} + c}$$

(b) (4 points) $\displaystyle\int 4xe^{x^2}\sqrt{e^{x^2}+1}\,dx$

Let $u = e^{x^2} + 1$

$du = 2xe^{x^2}dx \Rightarrow 2\,du = 4xe^{x^2}dx$

$$\rightsquigarrow \int 2\sqrt{u}\,du = \frac{4}{3}u^{3/2} + c \rightsquigarrow \boxed{\frac{4}{3}(e^{x^2}+1)^{3/2} + c}$$

(c) (4 points) $\displaystyle\int \frac{1}{8x-3}\,dx$

Let $u = 8x - 3$

$du = 8\,dx \Rightarrow \frac{1}{8}du = dx$

$$\rightsquigarrow \int \frac{1}{8} \cdot \frac{1}{u}\,du = \frac{1}{8}\ln|u| + c \rightsquigarrow \boxed{\frac{1}{8}\ln|8x-3| + c}$$

7. Evaluate the following definite integrals.

(a) (5 points) $\int_1^2 x\sqrt{x-1}\,dx$

Let $u = x - 1 \Rightarrow u + 1 = x$

$du = dx$

$$\rightsquigarrow \int_0^1 (u+1)\sqrt{u}\,du = \int_0^1 u^{3/2} + u^{1/2}\,du = \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\Big|_0^1 = \frac{2}{5} + \frac{2}{3} = \frac{16}{15}$$

(b) (5 points) $\int_1^e \frac{(\ln x)^3}{x}\,dx$

Let $u = \ln x$

$du = \frac{1}{x}\,dx$

$$\rightsquigarrow \int_0^1 u^3\,du = \frac{1}{4}u^4\Big|_0^1 = \frac{1}{4}$$

8. (6 points) A curve is defined by the equation

$$x^2y^2 - 2x = 4 - 4y.$$

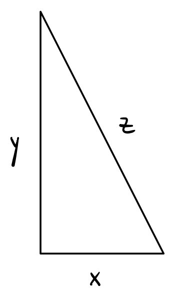
Give an equation for the tangent line to the curve at the point $(2, -2)$.

IMP. DIFF. $2xy^2 + 2x^2y\frac{dy}{dx} - 2 = -4\frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{2 - 2xy^2}{2x^2y + 4} \Rightarrow \frac{dy}{dx}\Big|_{(2,-2)} = \frac{2-16}{-16+4} = \frac{7}{6}$$

TANGENT LINE: $\boxed{y + 2 = \frac{7}{6}(x-2)}$ or $y = \frac{7}{6}x - \frac{13}{3}$

9. (6 points) A car is travelling 50 mph due south at a point 2 miles north of an intersection. Another car is traveling at 40 mph due west at a point 1 mile east of the same intersection. At what rate is the distance between the two cars changing?



GIVEN: $\frac{dx}{dt} = -40$, $\frac{dy}{dt} = -50$. FIND $\frac{dz}{dt}$.

$$x^2 + y^2 = z^2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$\therefore \frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z} = \frac{(1)(-40) + (2)(-50)}{\sqrt{5}} = \boxed{\frac{-140}{\sqrt{5}} \text{ mph}}$$

10. (6 points) A sample of radioactive material has an initial mass of 50 grams. After 20 days, the mass of the sample is 40 grams. Assuming that the mass of the material follows an exponential growth/decay model, find the half-life of the material.

$M(t) = 50\left(\frac{4}{5}\right)^{t/20}$. HALF-LIFE λ: $M(\lambda) = 50\left(\frac{4}{5}\right)^{\lambda/20} = 25$

$$\Rightarrow \left(\frac{4}{5}\right)^{\lambda/20} = \frac{1}{2} \Rightarrow \frac{\lambda}{20}\ln\left(\frac{4}{5}\right) = \ln\left(\frac{1}{2}\right) \Rightarrow \boxed{\lambda = \frac{20\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{4}{5}\right)}}$$

11. (6 points) Determine the absolute maximum and minimum values of

$$f(x) = 2x^3 + 3x^2 - 12$$

on the interval $[-1, 2]$.

crit #'s : $f'(x) = 6x^2 + 6x = 6x(x+1) = 0 \Rightarrow x = 0, -1$

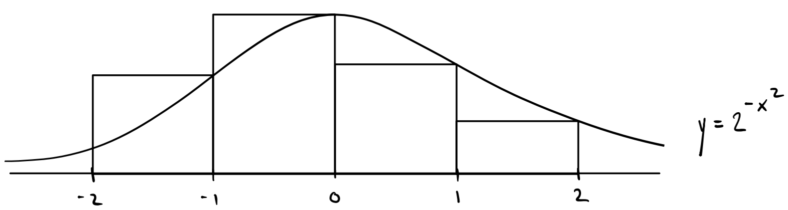
x	$f(x)$
-1	-11
0	-12
2	16

ABS. MAX VALUE $f(2) = 16$

ABS. MIN VALUE $f(0) = -12$

12. (6 points) Use a Riemann sum with 4 subintervals and right endpoints (R_4) to estimate the integral

$$\int_{-2}^{2} 2^{-x^2}\, dx.$$



$R_4 = f(-1)\Delta x + f(0)\Delta x + f(1)\Delta x + f(2)\Delta x$, $\Delta x = \frac{2-(-2)}{4} = 1$

$= 2^{-1} + 2^{0} + 2^{-1} + 2^{-4} = \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{16} = 2\frac{1}{16} = \frac{33}{16}$