Exam 1

1. (6 points) Use interval notaton to describe the domain of the function $f(x) = \dfrac{\sqrt{3x+4}}{x^2 - x}$.

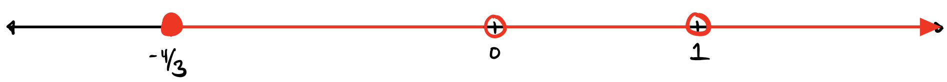
SINCE WE CANNOT TAKE SQUARE ROOT OF NEGATIVE NUMBERS, WE MUST HAVE

$$3x + 4 \geq 0 \implies \boxed{x \geq -\tfrac{4}{3}}$$

SINCE WE CANNOT DIVIDE BY 0, WE MUST HAVE

$$x^2 - x \neq 0 \implies x(x-1) \neq 0 \implies \boxed{x \neq 0 \text{ AND } x \neq 1}$$

PUTTING ALL OF THESE CONDITIONS TOGETHER, WE HAVE



$$\boxed{[-\tfrac{4}{3}, 0) \cup (0, 1) \cup (1, \infty)}$$

2. Let $f(x) = x^3 + 2x$ and let $g(x) = 1 - \sqrt{x}$.
   (a) (4 points) Find $f(g(x))$.
   (b) (4 points) Find $g(f(x))$.

(a) $f(g(x)) = f(1 - \sqrt{x}) = \boxed{(1-\sqrt{x})^3 + 2(1-\sqrt{x})}$

(b) $g(f(x)) = g(x^3 + 2x) = \boxed{1 - \sqrt{x^3 + 2x}}$

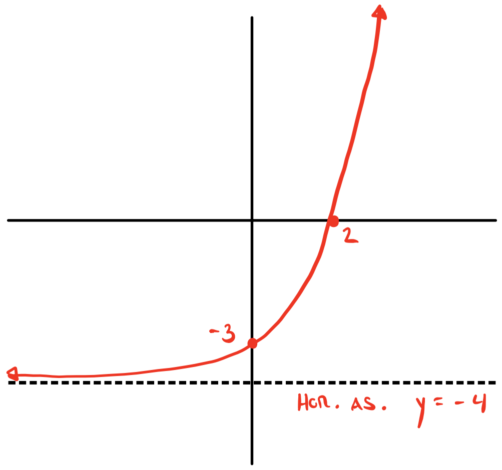
3. (6 points) Give an equation for the line that passes through the points $(-1, -2)$ and $(4, 3)$.

SLOPE $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3 - (-2)}{4 - (-1)} = \dfrac{3+2}{4+1} = 1$

USING POINT-SLOPE FORMULA $y - y_1 = m(x - x_1)$ WITH $(x_1, y_1) = (-1, -2)$ AND $m = 1$ GIVES

$$y + 2 = x + 1 \quad \text{OR} \quad \boxed{y = x - 1}$$

4. (6 points) Make a rough sketch of the graph $y = 2^x - 4$. Label the x-intercept, y-intercept, and horizontal asymptote.



X-INT: SET $y=0$, SOLVE FOR x

$0 = 2^x - 4 \Rightarrow 2^x = 4 \Rightarrow x = 2$

Y-INT: SET $x=0$, SOLVE FOR y

$y = 2^0 - 4 = 1 - 4 = -3$

5. (a) (4 points) Evaluate $\log_{16} 4$.
   (b) (6 points) Solve $\ln(x-1) = \ln(x) - 1$.

(a) WRITE $\log_{16} 4 = w$. THEN $16^w = 4$.

SINCE $16^{1/2} = \sqrt{16} = 4$, WE HAVE $w = \boxed{\frac{1}{2}}$

(b) $\ln(x-1) = \ln(x) - 1 \Rightarrow \ln(x-1) - \ln(x) = -1$

$\Rightarrow \ln\left(\frac{x-1}{x}\right) = -1 \Rightarrow \frac{x-1}{x} = e^{-1} = \frac{1}{e}$

$\Rightarrow x - 1 = \frac{1}{e}x \Rightarrow x - \frac{1}{e}x = 1 \Rightarrow x\left(1 - \frac{1}{e}\right) = 1$

$\Rightarrow \boxed{x = \frac{1}{1-\frac{1}{e}} = \frac{e}{e-1}}$

6. The population P of a bacteria t days after being placed in an ideal environment is given by an exponential function $P(t) = C \cdot a^t$, for some constants C and a.

   (a) (6 points) If the population after 5 days is 6,000 and the population after 10 days is 48,000, find the exponential function $P(t)$.

   (b) (4 points) When will the population reach 750,000? Leave your answer as a logarithmic expression.

(a) $P(t) = Ca^t$ $\qquad$ $P(5) = Ca^5 = 6000 \quad (1)$

$P(5) = 6{,}000$ $\qquad$ $P(10) = Ca^{10} = 48{,}000 \quad (2)$

$P(10) = 48{,}000$

DIVIDE (2) BY (1): $\dfrac{Ca^{10}}{Ca^5} = \dfrac{48{,}000}{6{,}000} \Rightarrow a^5 = 8 \Rightarrow \boxed{a = 8^{1/5}}$

NOW $P(5) = C(8^{1/5})^5 = 6{,}000 \Rightarrow 8C = 6{,}000$

$\Rightarrow \boxed{C = \dfrac{6{,}000}{8} = 750}$

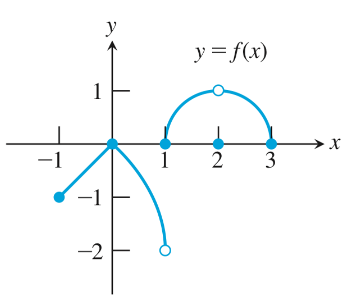
$\therefore \boxed{P(t) = 750(8^{1/5})^t = 750 \cdot 8^{t/5}}$

(b) SET $P(t) = 750 \cdot 8^{t/5} = 750{,}000$ & SOLVE FOR t

$8^{t/5} = 1{,}000 \Rightarrow \dfrac{t}{5}\ln(8) = \ln(1{,}000)$

$\Rightarrow t = \boxed{\dfrac{5\ln(1{,}000)}{\ln(8)} \text{ DAYS}}$

7. Use the graph below to answer the following questions. If a limit does not exist, write DNE.



(a) (2 points) $\lim_{x \to 1^-} f(x)$ $\boxed{= -2}$

(b) (2 points) $\lim_{x \to 1^+} f(x)$ $\boxed{= 0}$

(c) (2 points) $\lim_{x \to 1} f(x)$ $\boxed{\text{DNE}}$

(d) (4 points) Is f continuous at $x = 2$? Why or why not?

$\boxed{\text{No, } \underbrace{\lim_{x \to 2} f(x)}_{1} \neq \underbrace{f(2)}_{0}}$

8. Calculate each of the following limits. If a limit does not exist, write DNE.

(a) (6 points) $\displaystyle\lim_{x\to -1} \frac{x^2+2x+1}{x^4-1}$

(b) (6 points) $\displaystyle\lim_{x\to 1} \frac{\frac{1}{4}+\frac{1}{x}}{4+x}$

(a) $\displaystyle\lim_{x\to -1} \frac{(x+1)^{\cancel{2}}}{(x^2+1)\cancel{(x+1)}(x-1)} \overset{(x=-1)}{=} \frac{-1+1}{((-1)^2+1)(-1-1)} = \frac{0}{(2)(-2)} = \boxed{0}$

(b) Plug in $x=1$: $\displaystyle\frac{\frac{1}{4}+1}{4+1} = \frac{5/4}{5} = \boxed{\frac{1}{4}}$

9. Let $f(x) = \sqrt{16-x}$.

(a) (4 points) Find the average rate of change in f over the interval $[7, 12]$.

(b) (8 points) Find the instantaneous rate of change in f when $x = 7$. That is, find $f'(7)$.
Note: for this question you must calculate $f'(7)$ as a limit.

(a) $\displaystyle\frac{f(12)-f(7)}{12-7} = \frac{\sqrt{16-12}-\sqrt{16-7}}{12-7} = \frac{2-3}{12-7} = \boxed{\frac{-1}{5}}$

(b) $\displaystyle f'(7) = \lim_{x\to 7} \frac{f(x)-f(7)}{x-7} = \lim_{x\to 7} \frac{\sqrt{16-x}-3}{x-7} \cdot \frac{\sqrt{16-x}+3}{\sqrt{16-x}+3}$

$\displaystyle = \lim_{x\to 7} \frac{16-x-9}{(x-7)(\sqrt{16-x}+3)} = \lim_{x\to 7} \frac{7-x}{(x-7)(\sqrt{16-x}+3)} = \lim_{x\to 7} \frac{-\cancel{(x-7)}}{\cancel{(x-7)}(\sqrt{16-x}+3)}$

$\displaystyle = \lim_{x\to 7} \frac{-1}{\sqrt{16-x}+3} \overset{(x=7)}{=} \frac{-1}{\sqrt{16-7}+3} = \boxed{\frac{-1}{6}}$

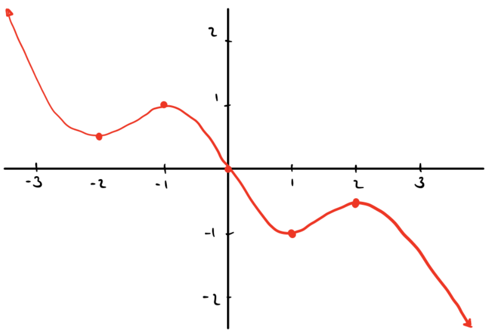
10. (8 points) Let $f(x) = 2x^2 + 5x$. Find $f'(x)$.
Note: for this question you must calculate $f'(x)$ as a limit.

$\displaystyle f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{2(x+h)^2+5(x+h)-(2x^2+5x)}{h}$

$\displaystyle = \lim_{h\to 0} \frac{\cancel{2x^2}+4xh+2h^2+\cancel{5x}+5h-\cancel{2x^2}-\cancel{5x}}{h} = \lim_{h\to 0} \frac{\cancel{h}(4x+2h+5)}{\cancel{h}}$

$\overset{(h=0)}{=} \boxed{4x+5}$

11. Use the graph of $f'(x)$ below to answer the following questions about $f(x)$.



(a) (4 points) On what interval(s) is $f(x)$ increasing/decreasing?

(b) (4 points) On what interval(s) is $f(x)$ concave up/down?

(c) (4 points) How many local minimums does $f(x)$ have?

(a) f IS INCREASING WHEN f' IS POSITIVE : $\boxed{\text{INCREASING } (-\infty, 0)}$

f IS DECREASING WHEN f' IS NEGATIVE : $\boxed{\text{DECREASING } (0, \infty)}$

(b) f IS CONCAVE UP WHEN f' IS INCREASING : $\boxed{\text{CONCAVE UP } (-2,-1) \cup (1,2)}$

f IS CONCAVE DOWN WHEN f' IS DECREASING : $\boxed{\text{CONCAVE DOWN } (-\infty,-2) \cup (-1,1) \cup (2,\infty)}$

(c) f HAS A LOCAL MINIMUM WHEN f CHANGES FROM DECREASING TO INCREASING,

i.e. WHEN f' CHANGES FROM NEGATIVE TO POSITIVE. THIS DOES NOT HAPPEN.

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