§5.4 The Substitution Rule

Recall the chain rule: $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\,g'(x)$.

This rule about derivatives can be phrased as a rule about antiderivatives, i.e. indefinite integrals:

$$\int f'(g(x))\,g'(x)\,dx = f(g(x)) + C$$

ex. Since $\frac{d}{dx}\left[e^{x^2}\right] = e^{x^2}\cdot 2x$

we have $\int e^{x^2}\cdot 2x = e^{x^2} + C$

IDEA (PATTERN)

$$\int f(\underbrace{g(x)}_{\text{inner function}})\,\underbrace{g'(x)\,dx}_{\text{derivative of inner function}} = F(g(x)) + C$$

where $F'(x) = f(x)$.

Method: Let $u = g(x)$. Then $\frac{du}{dx} = g'(x)$, which we write as $du = g'(x)\,dx$ (think: multiply both sides by dx).

Then $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$

$$= F(u) + C = F(g(x)) + C$$

where $F'(x) = f(x)$.

EXAMPLE 1 Find $\int x^2(x^3+2)^4\,dx$.

Let $u = x^3+2$

$du = 3x^2\,dx \Rightarrow \frac{1}{3}\,du = x^2\,dx$

$\underbrace{\qquad\qquad\qquad\qquad}$

You can manipulate these equations!

$$\therefore \int (x^3+2)^4 x^2\,dx = \int u^4 \cdot \frac{1}{3}\,du = \frac{1}{3}\int u^4\,du = \frac{1}{3}\cdot\frac{1}{5}u^5 + C$$

$$= \frac{1}{15}(x^3+2)^5 + C$$ (You can check this answer)

EXAMPLE 2 Find $\displaystyle\int \frac{dx}{(2x-4)^3}$. $= \int (2x-4)^{-3}\,dx$

Let $u = 2x-4$

$du = 2\,dx \Rightarrow \frac{1}{2}\,du = dx$

$$\therefore \int (2x-4)^{-3}\,dx = \int u^{-3}\cdot\frac{1}{2}\,du = \frac{1}{2}\int u^{-3}\,du = \frac{1}{2}\cdot\frac{1}{-2}u^{-2} + C$$

$$= -\frac{1}{4}(2x-4)^{-2} + C \quad \text{or} \quad \frac{-1}{4(2x-4)^2} + C$$

ex. Find $f(x)$ if $f'(x) = x^3(x^2+1)^{1/2}$ & $f(0) = 0$

f is an antiderivative of f'. The most general antiderivative of $f'(x)$ is

$\int f'(x)\,dx = \int x^3(x^2+1)^{1/2}\,dx$. Let $u = x^2+1 \Rightarrow u-1 = x^2$

$du = 2x\,dx \Rightarrow \frac{1}{2}\,du = x\,dx$

$$f(x) = \int (x^2+1)^{1/2} x^2 \, x\,dx = \int u^{1/2}(u-1)\frac{1}{2}\,du = \frac{1}{2}\int u^{3/2} - u^{1/2}\,du$$

$$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C = \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C.$$

Then $f(0) = \frac{1}{5} - \frac{1}{3} + C = 0 \Rightarrow C = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$

$$\therefore f(x) = \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + \frac{2}{15}.$$

DEFINITE INTEGRALS: 2 WAYS

(1) CHANGE LIMITS OF INTEGRATION WHEN PERFORMING SUBSTITUTION.

(2) FIND INDEFINITE INTEGRAL FIRST, THEN PLUG IN LIMITS OF INT. TO EVALUATE DEFINITE INTEGRAL.

EXAMPLE 7 Evaluate $\int_0^2 x\sqrt{x^2+4}\,dx$.

(1) Let $u = x^2 + 4$ $\longrightarrow$ WHEN $x = 2$ $\quad u = 2^2 + 4 = 8$

$du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$ $\qquad \int \dots dx \qquad \int \dots du$

WHEN $x = 0$ $\quad u = 0^2 + 4 = 4$

$$\therefore \int_0^2 \sqrt{x^2+4}\; x\,dx = \int_4^8 \sqrt{u} \cdot \frac{1}{2}\,du = \frac{1}{2}\int_4^8 u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_4^8$$

$$= \frac{1}{3}\left(8^{3/2} - 4^{3/2}\right)$$

(2) $$\underbrace{\int \sqrt{x^2+4}\; x\,dx}_{\text{INDEFINITE INTEGRAL}} = \int \sqrt{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$$

$$= \frac{1}{3}\left(x^2+4\right)^{3/2} + C$$

SO $$\int_0^2 \sqrt{x^2+4}\; x\,dx = \frac{1}{3}\left(x^2+4\right)^{3/2} + C\,\Big|_0^2$$

$$= \left(\frac{1}{3}\left(2^2+4\right)^{3/2} + C\right) - \left(\frac{1}{3}\left(0^2+4\right)^{3/2} + C\right)$$

$$= \frac{1}{3}(8)^{3/2} + C - \frac{1}{3}(4)^{3/2} - C = \frac{1}{3}\left(8^{3/2} - 4^{3/2}\right) \qquad \text{(SAME)}$$

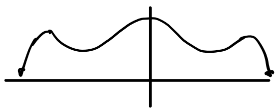
THEOREM 4 Integrals of Odd and Even Functions
Suppose that f is continuous on $[-a, a]$.

a. If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

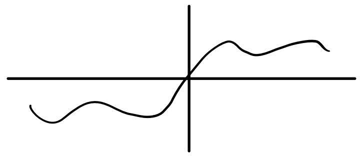
b. If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

Def:

EVEN: $f(-x) = f(x)$



ODD: $f(-x) = -f(x)$



ex. $\displaystyle\int_{-3}^{3} \underbrace{(x^2+2)}_{\text{EVEN}}\,dx = 2\int_{0}^{3} (x^2+2)\,dx$

ex. $\displaystyle\int_{-2}^{2} \underbrace{x^5 - x^3 + x}_{\text{ODD}}\,dx = 0.$

MORE ... (IGNORE ALL TRIG INTEGRALS)

7–30 Evaluate the indefinite integral.

7. $\int x\sqrt{1-x^2}\,dx$

8. $\int x^2\sin(x^3)\,dx$

9. $\int (1-2x)^9\,dx$

10. $\int \sin t\sqrt{1+\cos t}\,dt$

11. $\int \sin(2\theta/3)\,d\theta$

12. $\int \sec^2 2\theta\,d\theta$

13. $\int \sec 3t\tan 3t\,dt$

14. $\int y^2(4-y^3)^{2/3}\,dy$

15. $\int \cos(1+5t)\,dt$

16. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

17. $\int \sec^2\theta\tan^3\theta\,d\theta$

18. $\int \sin x\sin(\cos x)\,dx$

19. $\int (x^2+1)(x^3+3x)^4\,dx$

20. $\int x\sqrt{x+2}\,dx$

21. $\int \frac{a+bx^2}{\sqrt{3ax+bx^3}}\,dx$

22. $\int \frac{\cos(\pi/x)}{x^2}\,dx$

35–51 Evaluate the definite integral.

35. $\int_0^1 \cos(\pi t/2)\,dt$

36. $\int_0^1 (3t-1)^{50}\,dt$

37. $\int_0^1 \sqrt[3]{1+7x}\,dx$

38. $\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$

39. $\int_0^{\pi/6} \frac{\sin t}{\cos^2 t}\,dt$

40. $\int_{\pi/3}^{2\pi/3} \csc^2\left(\tfrac{1}{2}t\right)\,dt$

41. $\int_{-\pi/4}^{\pi/4} (x^3 + x^4\tan x)\,dx$

42. $\int_0^{\pi/2} \cos x\sin(\sin x)\,dx$

43. $\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}}$

44. $\int_0^a x\sqrt{a^2-x^2}\,dx$

45. $\int_0^a x\sqrt{x^2+a^2}\,dx \quad (a > 0)$

46. $\int_{-\pi/3}^{\pi/3} x^4\sin x\,dx$