§3.5 Implicit Differentiation

Application of the chain rule.

Warmup:

ex. Suppose f is differentiable. (f' exists)

Let $y = f(x)$. Let $w = x^2 f(x) = x^2 y$

Then $\frac{dw}{dx} = w' = \frac{d}{dx}[x^2] f(x) + x^2 \frac{d}{dx}[f(x)]$ Product rule

$= 2x f(x) + x^2 f'(x)$

$= 2xy + x^2 y'$

ex. Suppose f is differentiable. (f' exists)

Let $y = f(x)$. Let $w = x^2 f(x)^3 = x^2 y^3$

$\frac{dw}{dx} = w' = \frac{d}{dx}[x^2] f(x)^3 + x^2 \frac{d}{dx}[f(x)^3]$ Product rule

$= 2x f(x)^3 + x^2 \cdot 3f(x)^2 f'(x)$ Chain rule

$= 2xy^3 + 3x^2 y^2 y'$

IMPLICIT DIFFERENTIATION

CONSIDER THE EQUATION $x^2 + y^2 = 25$.

IT DESCRIBES A CURVE.

FIND EQ OF TANGENT LINE

AT THE POINT $(3,4)$.

IMPLICIT EQ.

NOT SOLVED FOR EITHER VARIABLE.

($y =$ ALL x'S ; $x =$ ALL y'S)

EXPLICIT

NOTE: THE IMPLICIT EQ DEFINES y AS ONE OR MORE FUNCTIONS OF x.

$$y = \begin{cases} \sqrt{25 - x^2} & \text{IF } y \geq 0 \\ -\sqrt{25 - x^2} & \text{IF } y < 0 \end{cases}$$

EXPLICIT EQ'S

$$\frac{dy}{dx} = \begin{cases} \frac{1}{2}(25 - x^2)^{-1/2}(-2x) & \text{IF } y \geq 0 \\ -\frac{1}{2}(25 - x^2)^{-1/2}(-2x) & \text{IF } y < 0. \end{cases}$$

Let's solve using **Implicit Differentiation**

(1) Assume that the implicit eq defines y as one or more differentiable functions of x, e.g. $y = f(x)$.

e.g. $x^2 + y^2 = 25 \longrightarrow x^2 + f(x)^2 = 25$

(2) Differentiate both sides of implicit eq:

$$\frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}\left[25\right] \qquad \frac{d}{dx}\left[x^2 + f(x)^2\right] = \frac{d}{dx}\left[25\right]$$

$$2x + 2y\,y' = 0 \qquad 2x + 2f(x)f'(x) = 0$$

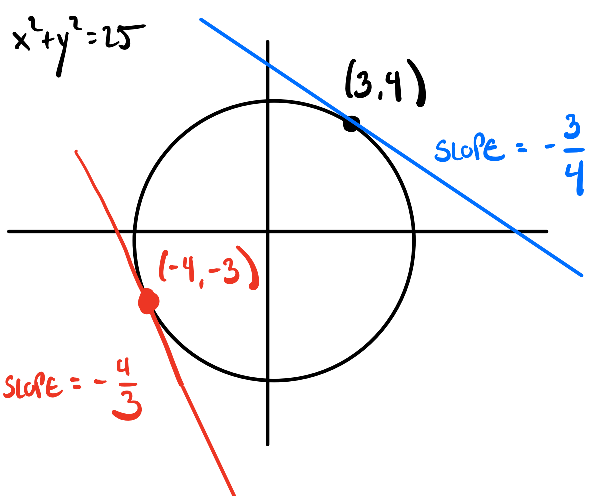
(3) Solve for y' $\left(\frac{dy}{dx}\right)$

$$2y\,y' = -2x$$

$$y' = -\frac{x}{y}$$

Check that this answer agrees with the old answer (explicit).

This gives the slope of the curve at <u>any</u> point (x,y) on the curve.



$y - 4 = -\frac{3}{4}(x-3)$

$y + 3 = -\frac{4}{3}(x+4)$

ex. $x^4 + x^2y^2 + y^2 = 4$. FIND y'.

$$\frac{d}{dx}\left[x^4 + x^2y^2 + y^2\right] = \frac{d}{dx}[4]$$

PRODUCT RULE + CHAIN RULE

$$\frac{d}{dx}\left[x^4 + x^2 f(x)^2 + f(x)^2\right] = \frac{d}{dx}[4]$$

ex. FIND y'' IF $x^4 + y^4 = 81$

$$4x^3 + 4y^3y' = 0 \implies y' = -\frac{x^3}{y^3}$$

$$y'' = -\frac{3x^2y^3 - x^3 \cdot 3y^2y'}{(y^3)^2} = \frac{-3x^2y^3 + 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6}$$

$$y'' = \frac{-3x^2y^3 - 3x^6y^{-1}}{y^6} = \frac{-3x^2(y^4 + x^4)}{y^7} = \frac{-243x^2}{y^7}$$

ADD'L EXERCISES:

11. $(x + 1)^2 + (y - 2)^2 = 9$

12. $(2x^2 + 3y^2)^{5/2} = x$

13. $\sqrt{x} + \sqrt{y} = 1$

14. $\sqrt{xy} = x^2 + 2y^2$

15. $y^2 = \sin(x + y)$

16. $x + y^2 = \cos xy$

17. $\tan^2(x^3 + y^3) = xy$

18. $x = \sec 2y$

19. $\sqrt{1 + \cos^2 y} = xy$

20. $x + y^2 = \cot xy$

34. (a) The curve with equation $y^2 = x^3 + 3x^2$ is called the **Tschirnhausen cubic**. Find an equation of the tangent line to this curve at the point $(1, -2)$.

(b) At what points does this curve have horizontal tangents?

(c) Illustrate parts (a) and (b) by graphing the curve and the tangent lines on a common screen.

$$\boxed{\frac{d}{dx}\left[\ln x\right] = \frac{1}{x}}$$

Proof: Let $y = \ln x$. Find y'

$$e^y = x$$

$$\frac{d}{dx}\left[e^y\right] = \frac{d}{dx}\left[x\right] \quad \text{(implicit differentiation)}$$

$$e^y \cdot y' = 1$$

$$y' = \frac{1}{e^y} = \frac{1}{x} \quad \checkmark$$

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ex. Find $\frac{d}{dx}\left[\ln(3x^5 - 5x^3 + 1)\right]$ $\longrightarrow$ $\boxed{\frac{d}{dx}\left[\ln(f(x))\right] = \frac{f'(x)}{f(x)}}$

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ex. Find $\frac{d}{dx}\left[x \ln(x^2 + 1)\right]$

Logarithmic Differentiation

1. Take ln of both sides.
2. Apply log laws
3. Perform implicit differentiation.

ex. Let $y = 2^{x^2 - x}$. Find y'.

ex. Let $y = \frac{e^x (3x+7)^8 (4x-3)^9}{(5x-2)^{10}}$