§3.3 Product & Quotient Rules

Product Rule

$$\frac{d}{dx}\left[f(x)\, g(x) \right] = f'(x)\,g(x) + f(x)\,g'(x)$$

Proof:

$$\lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x+h) + f(x)\,g(x+h) - f(x)\,g(x)}{h}$$

$$= \lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h} \cdot g(x+h) + f(x)\,\frac{g(x+h)-g(x)}{h} \right]$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \cdot \lim_{h\to 0} g(x+h) + f(x) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= f'(x)\,g(x) + f(x)\,g'(x) \quad \blacksquare$$

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ex. Check that $\underbrace{\frac{d}{dx}\left[x^8\right]}_{\text{Power Rule}} = \underbrace{\frac{d}{dx}\left[(x^3)(x^5)\right]}_{\text{Product Rule}}$

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ex. Find derivative of $F(x) = \frac{1}{3}e^x\left(4x^3 - 3x^2\right)$

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ex. Find derivative of $F(x) = \dfrac{e^x + 5x}{\sqrt{x}}$

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ex. Find $\frac{d}{dx}\left[(2 + 3x^2)(x^3 - 5)\right]$ (Product rule OR distribute first)

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g(x)^2}$$

Proof: $\lim_{h \to 0} \frac{1}{h}\left(\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right)$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{f(x+h)\,g(x) - f(x)\,g(x+h)}{g(x+h)\,g(x)}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\left(f(x+h)\,g(x) - f(x)\,g(x)\right) - \left(f(x)\,g(x+h) - f(x)\,g(x)\right)}{g(x+h)\,g(x)}\right]$$

$$= \lim_{h \to 0} \frac{g(x)}{g(x+h)\,g(x)} \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h \to 0} \frac{f(x)}{g(x+h)\,g(x)} \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \frac{g(x)}{g(x)^2}\,f'(x) - \frac{f(x)}{g(x)^2}\,g'(x) = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g(x)^2} \quad \blacksquare$$

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ex. Verify that $\frac{d}{dx}\left[x^5\right] = \frac{d}{dx}\left[\frac{x^8}{x^3}\right]$

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ex. Find $\frac{d}{dx}\left[\frac{9x - 7}{3 - 5x}\right]$

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ex. Find $\frac{d}{dx}\left[\frac{x^4 - 5x^3}{3e^x}\right]$

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ex. Find $\frac{d}{dx}\left[\frac{4e^x + 7x^2}{\sqrt{x}}\right]$

Note: If we split the fraction, we can use Product & Power rule instead of Quotient rule.
