Symmetric rigidity for circle endomorphisms with bounded geometry and their dual maps

John Adamski

The Graduate Center, CUNY

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Outline

Circle homeomorphisms

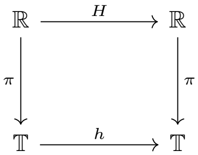
Let $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ and $\pi(x) = e^{2\pi i x} : \mathbb{R} \to \mathbb{T}$.

Definition

A **circle homeomorphism** is an orientation preserving homeomorphism $h : \mathbb{T} \to \mathbb{T}$ with lift $H : \mathbb{R} \to \mathbb{R}$ satisfying

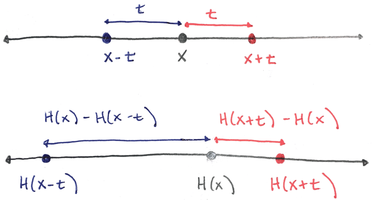
$$\pi \circ H(x) = h \circ \pi(x)\ ,\ H(x+1) = H(x) + 1,\ \forall x \in \mathbb{R}.$$

We assume that $0 \le H(0) < 1$.



h is **quasisymmetric** if $\exists M > 1$ such that

$$\frac{1}{M} \leq \frac{H(x+t) - H(x)}{H(x) - H(x-t)} \leq M \ , \ \forall x \in \mathbb{R} \ , \ \forall t > 0.$$



h is **symmetric** if $\exists \epsilon : \mathbb{R} \to \mathbb{R}$ such that $\epsilon(t) \to 0^+$ as $t \to 0^+$ and

$$\frac{1}{1+\epsilon(t)} \leq \frac{H(x+t) - H(x)}{H(x) - H(x-t)} \leq 1 + \epsilon(t) \ , \ \forall x \in \mathbb{R} \ , \ \forall t > 0.$$

Circle endomorphisms

Definition

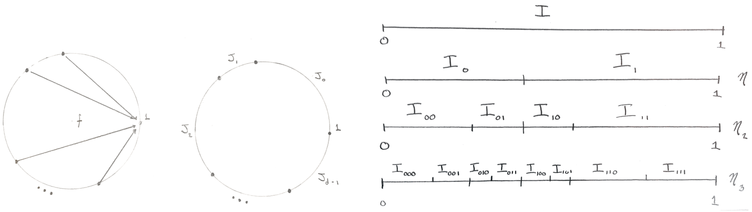
A **circle endomorphism** is an orientation preserving covering map $f : \mathbb{T} \to \mathbb{T}$ of topological degree $d \geq 2$. We assume $f(1) = 1$. The lift $F : \mathbb{R} \to \mathbb{R}$ satisfies

$$\pi \circ F(x) = f \circ \pi(x) \ , \ F(x+1) = F(x) + d, \ \forall x \in \mathbb{R}.$$

We assume $F(0) = 0$.

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{T} & \xrightarrow{f} & \mathbb{T} \end{array}$$

Nested Markov partitions for circle endomorphisms



- $f^{-1}(1)$ creates $\xi_1 = \{J_0, J_1, \dots, J_{d-1}\}$.
- Lift to partition of $[0,1]$, $\eta_1 = \{I_0, I_1, \dots, I_{d-1}\}$.
- Pull-back partitions: $\xi_n = f^{-(n-1)}(\xi_1)$.

$$\xi_n = \{J_{\omega_n} \mid \omega_n = i_0 i_1 \dots i_{n-1},\ i_k \in \{0,1,\dots,d-1\},\ 0 \le k \le d-1\}.$$

$$f^k(J_{\omega_n}) \in J_{i_k} \text{ for } 0 \le k \le n-1$$

- I_{ω_n} is the lift of J_{ω_n} to $[0,1]$.
- **Given $\{\eta_n\}_{n=1}^{\infty}$ and $0 < C \le \tau < 1$ with $C^n \le |I_{\omega_n}| \le \tau^n$, there is a unique circle endomorphism f that creates it.**

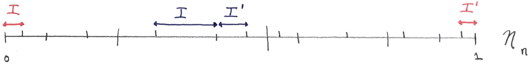
Definition

1. f is **uniformly quasisymmetric** if $\exists M > 1$ such that

$$\frac{1}{M} \leq \frac{F^{-n}(x+t) - F^{-n}(x)}{F^{-n}(x) - F^{-n}(x-t)} \leq M \ , \ \forall x \in \mathbb{R} \ , \ \forall t > 0 \ , \ \forall n \geq 1.$$

2. f has **bounded nearby geometry** if $\exists M \geq 1$ such that $\forall n \geq 1$ and $\forall I, I' \in \eta_n$ that share an endpoint (modulo 1),

$$\frac{1}{M} \leq \frac{|I'|}{|I|} \leq M.$$



Theorem

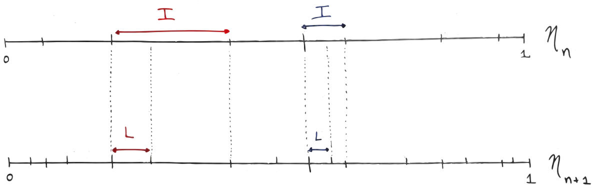
(1) $\iff$ *(2)* $\iff$ $f = h \circ q_d \circ h^{-1}$, h *quasisymmetric.*

Bounded geometry (weaker condition)

Definition

f has **bounded geometry** if $\exists 0 < C \leq \tau(C) < 1$ such that

$$C \leq \frac{|L|}{|I|} < \tau, \quad \forall L \subset I,\ I \in \eta_n,\ L \in \eta_{n+1},\ \forall n \geq 0.$$



Lemma

BNG $\Rightarrow$ *BG.*

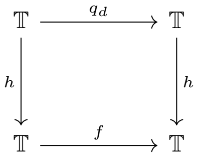
Topological conjugacies

- Any two circle endomorphisms f and g of the same degree $d \geq 2$ that both have **BG** are topologically conjugate.

$$f \circ h = h \circ g$$

  If f and g both have **BNG** then h is quasisymmetric.
- All circle endomorphisms f with **BG** are topologically conjugate to

$$q_d(z) = z^d, \qquad Q_d(x) = dx,$$



  and semi-conjugate to the **left-shift map** $\sigma : \Sigma \to \Sigma$.

Symbolic space Σ (review)

$$\Sigma = \prod_{n=0}^{\infty} \{0, 1, \ldots, d-1\}.$$

- A **point** $\omega \in \Sigma$ is an infinite sequence $\omega = i_0 i_1 i_2 \ldots$.
- A **left-cylinder** $[\omega_n] \subset \Sigma$ of length n is the set of all points whose first n terms agree with ω.
- The left-cylinders generate the **left topology**. The set Σ with the left topology is called the **symbolic space**.
- Then for a point $\omega = i_0 i_1 \ldots \in \Sigma$, we have

$$\Sigma \supset [\omega_1] \supset [\omega_2] \supset \cdots \supset [\omega_n] \supset \ldots,$$

just as

$$\mathbb{T} \supset J_{\omega_1} \supset J_{\omega_2} \supset \ldots \supset J_{\omega_n} \supset \ldots.$$

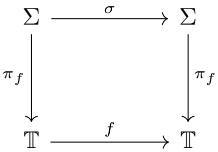
Symbolic dynamical system (review)

We define the **projection** $\pi_f : \Sigma \to \mathbb{T}$ as

$$\pi_f(\omega) = J_\omega = \cap_{n=1}^{\infty} J_{\omega_n} = x_\omega,$$

and the **left shift map** $\sigma : \Sigma \to \Sigma$,

$$\sigma : i_0 i_1 i_2 \cdots = i_1 i_2 \ldots .$$



(Σ, σ) is called the **symbolic dynamical system**. It is the space of all **forward orbits** for the dynamical system $(\mathbb{T}, f)$.

Borel probability measures on $\mathbb{T}$

- Let $\mathbb{T} = [0, 1]$ with $0 \sim 1$.
- Let $\mathcal{B}$ denote the Borel σ-algebra of subsets of $\mathbb{T}$.
- $M(\mathbb{T})$ denotes the set of all Borel probability measures μ on $\mathbb{T}$.
- λ and $|\cdot|$ denote the Lebesgue probability measure on $\mathbb{T}$.
- Non-atomic measures $\mu \in M(\mathbb{T})$ with full support are in one to one correspondence with circle homeomorphisms h.

$$h(x) = \mu([0, x]), \quad \forall x \in \mathbb{R} \qquad \text{(distribution function)}$$

$$\mu(A) = \lambda(h(A)), \quad \forall A \in \mathcal{B}; \qquad \mu = h^* \lambda \qquad \text{(pullback)}$$

Definition

The measure $\mu \in M(\mathbb{T})$ is **invariant w.r.t.** $f : \mathbb{T} \to \mathbb{T}$ if

$$\mu(f^{-1}(A)) = \mu(A), \qquad \forall A \in \mathcal{B}.$$

Equivalently, let $f_* : M(\mathbb{T}) \to M(\mathbb{T})$ such that

$$f_*(\mu)(A) = \mu(f^{-1}(A)), \quad \forall A \in \mathcal{B}(\mathbb{T}). \qquad \text{(pushforward)}$$

An invariant measure with respect to f is a fixed point of the map f_*.

Remark

Let h be circle homeomorphism and

$$\mu = h^*\lambda, \qquad f = h \circ \circ q_d \circ h^{-1}.$$

Then $\mu = h^*\lambda$ is invariant w.r.t. $q_d \iff \lambda$ is invariant w.r.t. $f = h \circ q_d \circ h^{-1}$.

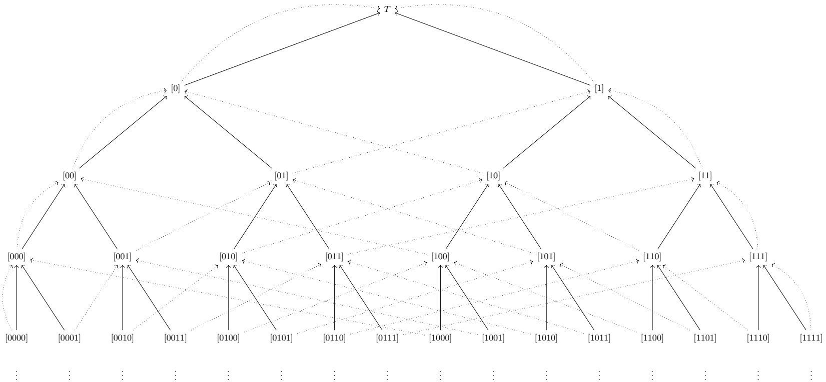


Figure: Two directed d-nary trees ($d = 2$). The dotted tree shows the preimages of each partition interval and the solid tree shows the subsets of each partition interval .

Two binary trees: Lebesgue measure λ preserved by both

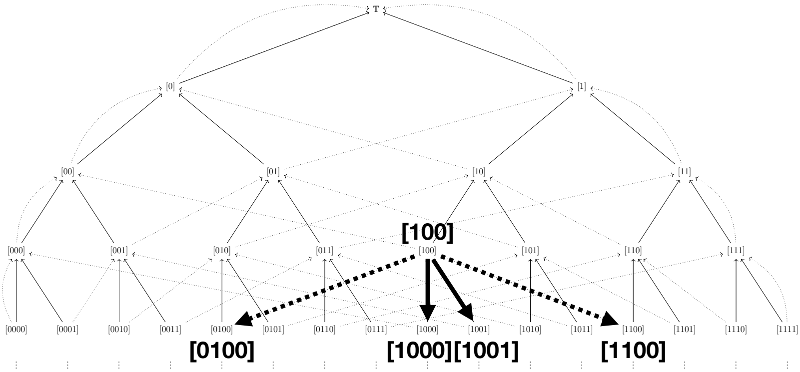


Figure: Every interval I_{o_n} has two subintervals $I_{\omega_n 0}, I_{\omega_n 1} \in \eta_{n+1}$ and two preimage intervals $I_{0\omega_n}, I_{1\omega_n} \in \eta_{n+1}$.

Swapping forward paths and backward paths (1/3)

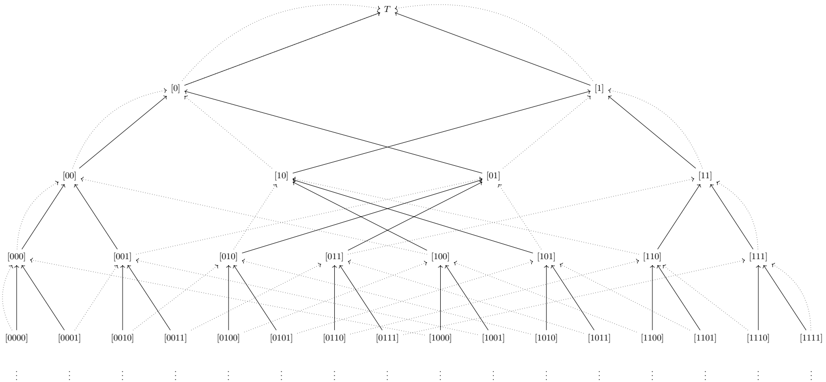


Figure: Since f preserves Lebesgue measure, the intervals of each partition η_n can be "shuffled" by swapping intervals whose labels are reversals of one another, and they still fit together as nested partitions. This is "untangling" the dotted tree.

Swapping forward paths and backwad paths (2/3)

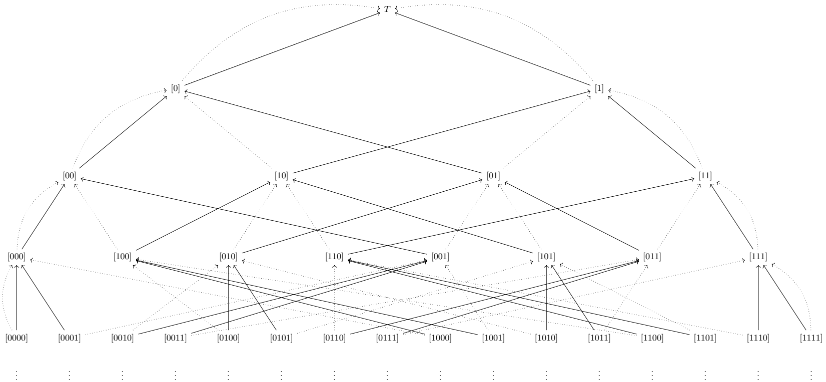


Figure: Since f preserves Lebesgue measure, the intervals of each partition η_n can be "shuffled" by swapping intervals whose labels are reversals of one another, and they still fit together as nested partitions. This is "untangling" the dotted tree.

Swapping forward paths and backwad paths (3/3)

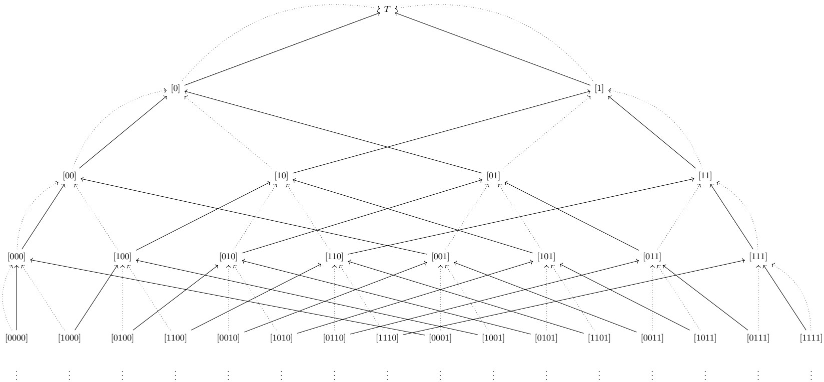
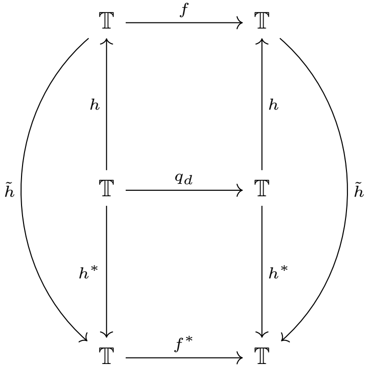


Figure: This yields a new sequence of nested partitions $\{\eta_n^*\}_{n=1}^{\infty}$ which defines a new circle endomorphism f^* that we call the **dual circle endomorphism**.

The dual conjugacy $\tilde{h}$

The circle endomorphism $f = h \circ q_d \circ h^{-1}$ and the dual circle endomorphism $f^* = h^* \circ q_d \circ (h^*)^{-1}$ are topologically conjugate.

$$f^* = \tilde{h} \circ f \circ \tilde{h}^{-1}$$
$$\tilde{h} = h^* \circ h^{-1}$$



Randomly generated dual circle endomorphisms and their dual conjugacy

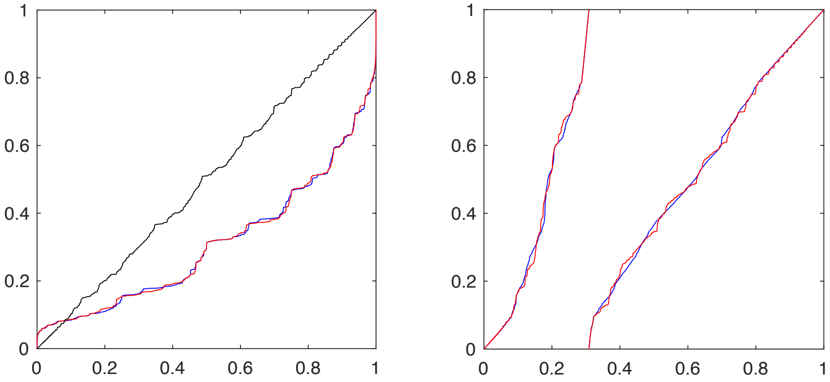


Figure: $C = .1$. LEFT: blue is h, red is h^*, black is $\tilde{h} = h^* \circ h^{-1}$. RIGHT: blue is $f = h \circ q_2 \circ h^{-1}$, red is $f^* = h^* \circ q_2 \circ (h^*)^{-1}$.

Properties of $\tilde{h}$, $d = 2$

- $\forall n \geq 0$, $\tilde{h}$ sends endpoints of intervals in η_n to the endpoints of intervals in η_n^*, preserving their order.
- I_0 and I_1 do not move $\Rightarrow h(0) = 0$, $h(1/2)$, and $h(1) = 1$ are fixed points.
- Furthermore,

$$|I_0| = |I_{00}| + |I_{10}| = |I_{00}| + |I_{01}|$$

$$\Rightarrow |I_{01}| = |I_{10}|$$

- There exist many fixed point of $\tilde{h}$.

Properties of $\tilde{h}$, $d = 2$

More generally, when subsets are also preimages and one endpoint is fixed,

$$|I_{\underbrace{0\dots0}_{n-1}}| = |I_{0\underbrace{0\dots0}_{n-1}}| + |I_{1\underbrace{0\dots0}_{n-1}}| = |I_{\underbrace{0\dots0}_{n-1}0}| + |I_{\underbrace{0\dots0}_{n-1}1}|$$

$$\Rightarrow |I_{1\underbrace{0\dots0}_{n-1}}| = |I_{\underbrace{0\dots0}_{n-1}1}|, \text{ and}$$

$$|I_{\underbrace{1\dots1}_{n-1}}| = |I_{0\underbrace{1\dots1}_{n-1}}| + |I_{1\underbrace{1\dots1}_{n-1}}| = |I_{\underbrace{1\dots1}_{n-1}0}| + |I_{\underbrace{1\dots1}_{n-1}1}|$$

$$\Rightarrow |I_{0\underbrace{1\dots1}_{n-1}}| = |I_{\underbrace{1\dots1}_{n-1}0}|.$$

That is,

$$\mathrm{Fix}(\tilde{h}) \supseteq \left\{ h\left(\frac{1}{2^n}\right), h\left(\frac{1}{2} \pm \frac{1}{2^n}\right), h\left(1 - \frac{1}{2^n}\right) \middle| 1 \leq n \leq N \right\}.$$

More fixed points of $\tilde{h}$

- for any $n \geq 1$, the endpoints of the interval

$$I_{\underbrace{0\ldots0}_{n}\,1}$$

are fixed by $\tilde{h}$. Let a_1 denote the left endpoint of this interval (fixed).

- Now consider the interval

$$I_{\underbrace{0\ldots0}_{n}\,1\,\underbrace{0\ldots0}_{n}}.$$

Note that the label of this interval is a palindrome $\Rightarrow$ its length is preserved $\Rightarrow$ its right endpoint a_2 is fixed.

- Note:

$$a_2 = \text{left endpoint of } I_{\underbrace{0\ldots0}_{n}\,1\,\underbrace{0\ldots0}_{n-1}\,1}.$$

More fixed points of $\tilde{h}$

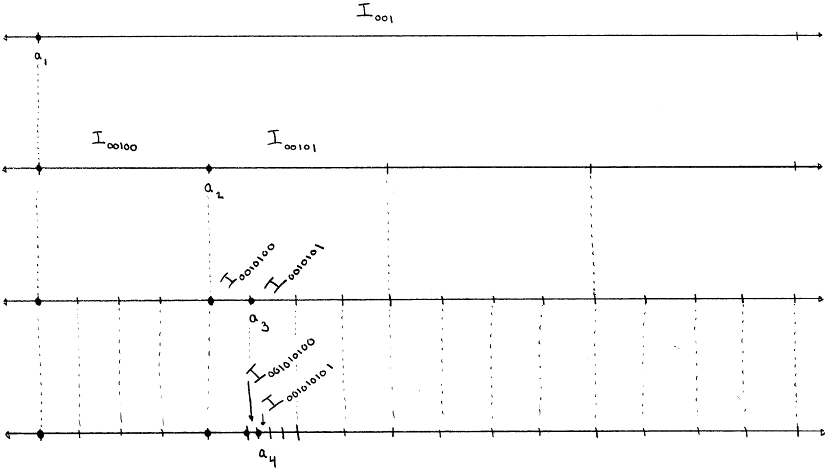


Figure: The increasing sequence $\{a_m\}_{m=1}^{\infty}$ of fixed points of $\tilde{h}$, constructed with $n = 2$.

More fixed points of $\tilde{h}$

In this way, define

$$a_3 = \text{left endpoint of } I_{\underbrace{0\ldots0}_{n}1\underbrace{0\ldots0}_{n-1}1\underbrace{0\ldots0}_{n-1}1}$$

$$a_4 = \text{left endpoint of } I_{\underbrace{0\ldots0}_{n}1\underbrace{0\ldots0}_{n-1}1\underbrace{0\ldots0}_{n-1}1\underbrace{0\ldots0}_{n-1}1}$$

$$\vdots$$

$$a_m = \text{left endpoint of } I_{\underbrace{0\ldots0}_{n}1\underbrace{\underbrace{0\ldots0}_{n-1}1\underbrace{0\ldots0}_{n-1}1\ldots\underbrace{0\ldots0}_{n-1}1}_{m-1}}.$$

$$a_m = h\left(\frac{1}{2^{n+1}} + \frac{1}{2^{2n+1}} + \frac{1}{2^{3n+1}} + \ldots \frac{1}{2^{mn+1}}\right).$$

Theorem

Suppose f is a degree 2 circle endomorphism with bounded geometry that preserves Lebesgue measure and f^* is its dual circle endomorphism. Let h and h^* be circle homeomorphisms such that

$$f = h \circ q_2 \circ h{-}1, \qquad f^* = h^* \circ q_2 \circ (h^*)^{-1},$$

and

$$\tilde{h} = h^* \circ h^{-1}, \qquad f^* = \tilde{h} \circ f \circ \tilde{h}^{-1}.$$

Then for all $n, m \geq 1$, there is a fixed point

$$x_{n,m} = h\left(\sum_{i=1}^{m} \frac{1}{2^{in+1}}\right) \in \mathrm{Fix}(\tilde{h}),$$

and for all $n \geq 1$, there is a limit point of fixed points

$$x_n = \lim_{m\to\infty} x_{n,m} = h\left(\sum_{i=1}^{\infty} \frac{1}{2^{in+1}}\right) = h\left(\frac{1}{2(2^n - 1)}\right) \in \overline{\mathrm{Fix}(\tilde{h})}.$$

A fixed point corollary for the dual conjugacy $\tilde{h}$

Switching all 0's to 1's and all 1's to 0's...

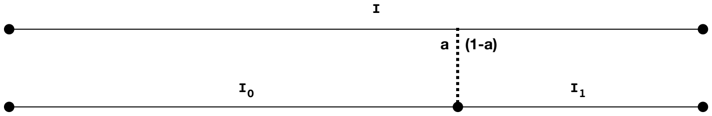
Corollary

For all $n, m \geq 1$, there is a fixed point

$$y_{n,m} = h\left(1 - \sum_{i=1}^{m} \frac{1}{2^{in+1}}\right) \in \mathrm{Fix}(\tilde{h}),$$

and for all $n \geq 1$, there is a limit point of fixed points

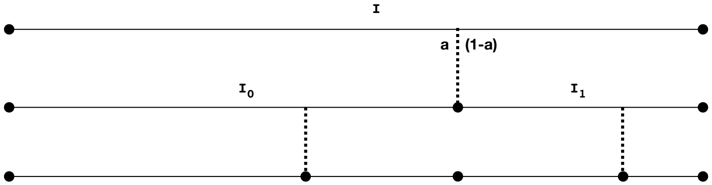
$$y_n = \lim_{m \to \infty} y_{n,m} = h\left(1 - \sum_{i=1}^{\infty} \frac{1}{2^{in+1}}\right) = h\left(1 - \frac{1}{2(2^n - 1)}\right) \in \overline{\mathrm{Fix}(\tilde{h})}.$$



$0 < C \leq \alpha \leq \tau < 1$ is the ratio for how to partition I.

$$\mu(I_0) = \alpha\mu(I), \quad \mu(I_1) = (1-\alpha)\mu(I)$$

Algorithm for constructing a measure with BG and invariant w.r.t. q_2

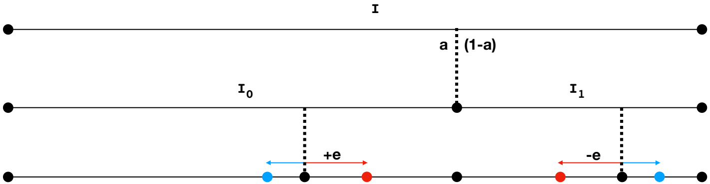


α_0 and α_1 will be the ratios for how to partition I_0 and I_1, respectively.

Measure must be preserved.

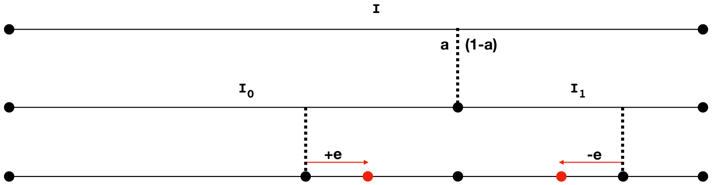
BG must be satisfied.

$$0 < C \leq \alpha_0, \alpha_1 \leq \tau < 1$$



$$\mu(I_{00}) = \alpha\mu(I_0) + \epsilon \qquad \mu(I_{10}) = \alpha\mu(I_1) - \epsilon$$

Algorithm for constructing a measure with BG and invariant w.r.t. q_2



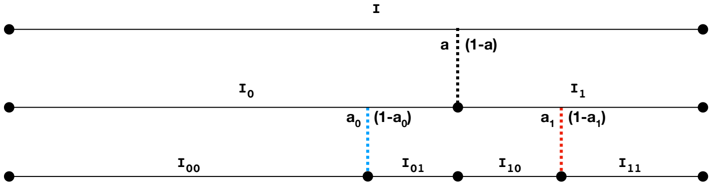
$$\mu(I_{00}) = \alpha\mu(I_0) + \epsilon := \alpha_0(I_0)$$

$$\Rightarrow \alpha_0 = \alpha + \frac{\epsilon}{\mu_1(I_0)}$$

$$\mu(I_{10}) = \alpha\mu(I_1) - \epsilon := \alpha_1(I_1)$$

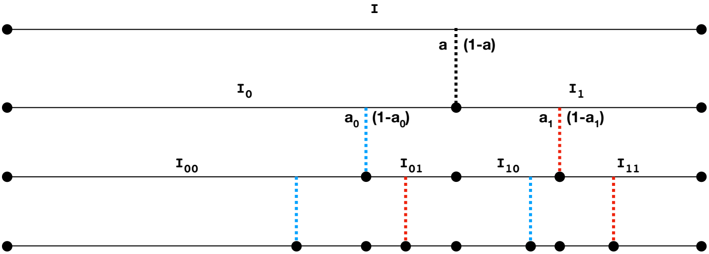
$$\Rightarrow \alpha_1 = \alpha - \frac{\epsilon}{\mu_1(I_1)}$$

Algorithm for constructing a measure with BG and invariant w.r.t. q_2



$$\mu(I_{00}) = \alpha_0(I_0) \qquad \mu(I_{10}) = \alpha_1(I_1)$$

$$\mu(I_{01}) = (1-\alpha_0)(I_0) \qquad \mu(I_{11}) = (1-\alpha_1)(I_1)$$



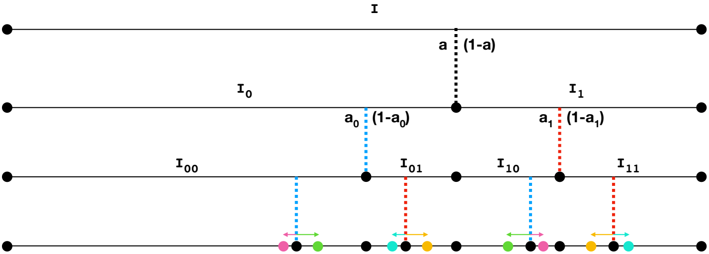
α_{ω_n} will be the ratio for how to partition I_{ω_n}.

Measure must be preserved.

BG must be satisfied.

$$0 < C \leq \alpha_{\omega_n} \leq \tau < 1$$

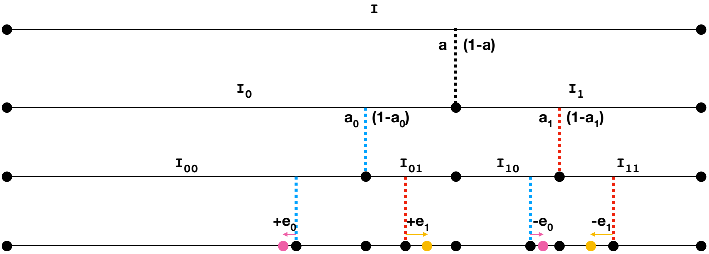
Algorithm for constructing a measure with BG and invariant w.r.t. q_2



$$C \leq \alpha_{00} = \alpha_0 + \frac{\epsilon_0}{\mu(I_{00})} \leq \tau, \qquad C \leq \alpha_{10} = \alpha_0 - \frac{\epsilon_0}{\mu(I_{10})} \leq \tau,$$

$$C \leq \alpha_{01} = \alpha_1 + \frac{\epsilon_1}{\mu(I_{01})} \leq \tau, \qquad C \leq \alpha_{11} = \alpha_1 - \frac{\epsilon_1}{\mu(I_{11})} \leq \tau.$$

Algorithm for constructing a measure with BG and invariant w.r.t. q_2



$$\mu(I_{000}) = \alpha_0\mu(I_{00}) + \epsilon_0 := \alpha_{00}\mu(I_{00})$$

$$\Rightarrow \alpha_{00} = \alpha_0 + \frac{\epsilon_0}{\mu(I_{00})}$$

$$\mu(I_{010}) = \alpha_1\mu(I_{01}) + \epsilon_1 := \alpha_{01}\mu(I_{01})$$

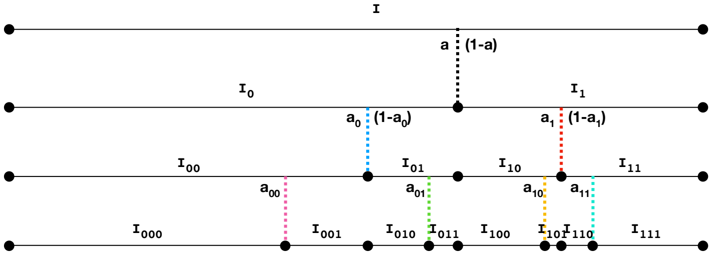
$$\Rightarrow \alpha_{01} = \alpha_1 + \frac{\epsilon_1}{\mu(I_{01})}$$

$$\mu(I_{100}) = \alpha_0\mu(I_{10}) - \epsilon_0 := \alpha_{10}\mu(I_{10})$$

$$\Rightarrow \alpha_{10} = \alpha_0 - \frac{\epsilon_0}{\mu(I_{10})}$$

$$\mu(I_{110}) = \alpha_1\mu(I_{11}) - \epsilon_1 := \alpha_{11}\mu(I_{11})$$

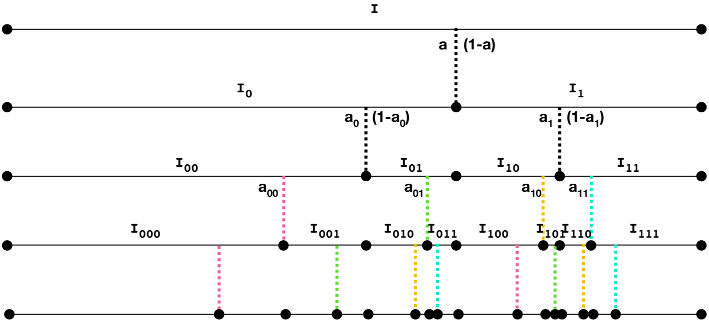
$$\Rightarrow \alpha_{11} = \alpha_1 - \frac{\epsilon_1}{\mu(I_{11})}$$



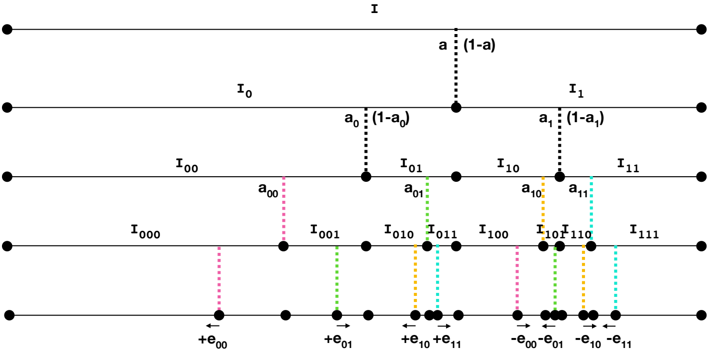
α_{ω_n} is the ratio for how to partition I_{ω_n}.

$$\mu(I_{\omega_n 0}) = \alpha_{\omega_n}\mu(I_{\omega_n}), \quad \mu(I_{\omega_n 1}) = (1 - \alpha_{\omega_n})\mu(I_{\omega_n})$$

Algorithm for constructing a measure with BG and invariant w.r.t. q_2



Algorithm for constructing a measure with BG and invariant w.r.t. q_2



$$C \leq \alpha_{0\omega_n} = \alpha_{\omega_n} + \frac{\epsilon_{\omega_n}}{\mu(I_{0\omega_n})} \leq \tau, \qquad C \leq \alpha_{1\omega_n} = \alpha_{\omega_n} - \frac{\epsilon_{\omega_n}}{\mu(I_{1\omega_n})} \leq \tau.$$

$$-(a_{\omega_n} - C)\mu(I_{0\omega_n}) \leq \epsilon_{\omega_n} \leq (\tau - \alpha_{\omega_n})\mu(I_{1\omega_n})$$
$$-(\tau - \alpha_{\omega_n})\mu(I_{0\omega_n}) \leq \epsilon_{\omega_n} \leq (a_{\omega_n} - C)\mu(I_{1\omega_n}).$$

Randomly generated circle endomorphism

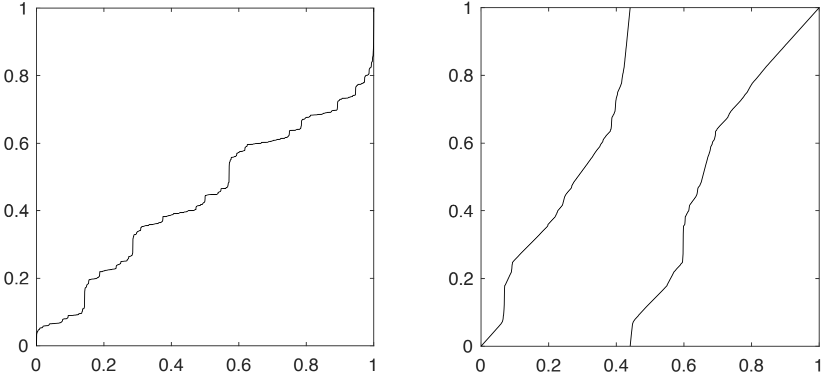


Figure: LEFT: distribution function $h(x) = \mu([0, x])$ for a randomly generated measure μ that is invariant w.r.t q_2 and has BG with $C = .1$. RIGHT: $f = h \circ q_2 \circ h^{-1}$.

Scatter plots of α_{ω_n} values above the midpoints of intervals I_{ω_n,q_d}.

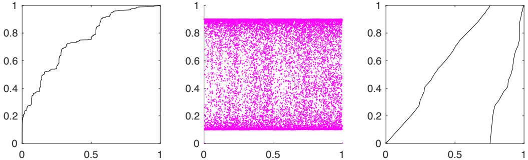


Figure: $C = .1$, $\tau = .9$.

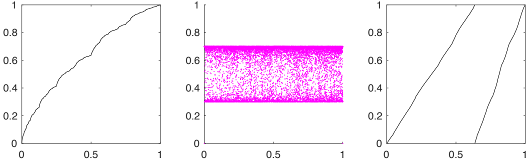


Figure: $C = .3$, $\tau = .7$.

α_{ω_n} values define a sequence of simple functions on dual symbolic space Σ^*

- A **point** $\omega \in \Sigma^*$ is an infinite sequence $\omega^* = \dots j_2 j_1 j_0$.
- A **left-cylinder** $[\omega_n^*] \subset \Sigma$ of length n is the set of all points whose first n terms (*on the right*) agree with ω^*.
- These generate the **right topology** and the Borel sigma algebra $\mathcal{B}^*$
- Define a probability measure $P([\omega_n^*]) = |I_{\omega_n^*}|$.
- Define the **right shift map** $\sigma^* : \Sigma^* \to \Sigma^*$

$$\sigma^* : \dots j_2 j_1 j_0 \to \dots j_2 j_1.$$

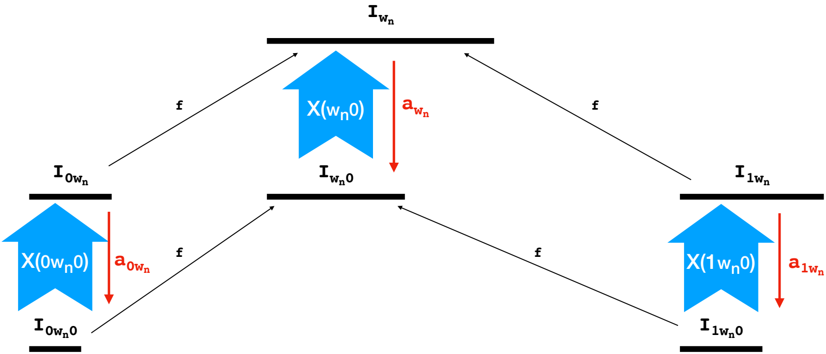
Define $\mathcal{B}_n^*$ as the σ-algebra generated by all right-cylinders of length n, and

$$X_n(w^*) = \frac{P([\sigma^*(w_n^*)])}{P([w_n^*])} = \frac{|I_{\sigma^*(w_n^*)}|}{|I_{w_n^*}|}.$$

That is,

$$X_{n+1}(\omega 0^*) = \frac{1}{\alpha_{\omega_n}}, \qquad X_{n+1}(\omega 1^*) = \frac{1}{1-\alpha_{\omega_n}}$$

α_{ω_n} values define a Martgingale on the dual symbolic space



$$|I_{0\omega_n}| + |I_{1\omega_n}| = |I_{\omega_n}|$$

$$X_{n+2}(\ldots 0\omega_n 0)|I_{0\omega_n 0}| + X_{n+2}(\ldots 0\omega_n 0)|I_{0\omega_n 0}| = X_{n+1}(\ldots \omega_n 0)|I_{\omega_n 0}|$$

Theorem

The sequence $(X_n, \mathcal{B}_n^*)$ *is a martingale on* $(\Sigma^*, \mathcal{B}^*, P)$. *That is,*

1. $E[|X_n|] < \infty$ *for all* $n \geq 0$.
2. X_n *is* $\mathcal{B}_n^*$*-measurable for all* $n \geq 1$.
3. *For all* $1 \leq m \leq n$, $X_m = E[X_n | \mathcal{B}_m^*]$ $P - a.e.$,

Theorem

Suppose $f : T \to T$ *is a circle endomorphism with bounded geometry that preserves* λ. *Let*

$$X_n(w^*) = \frac{P([\sigma^*(w_n^*)])}{P([w_n^*])} = \frac{|I_{\sigma^*(w_n^*)}|}{|I_{w_n^*}|}.$$

Then there exists a bounded $\mathcal{B}^*$*-measurable function* $X \in L^1(P)$ *such that*

1. $\lim_{n\to\infty} X_n = X(= X_f)$ *exists* $P - a.e.$,
2. $\lim_{n\to\infty} \int_{\Sigma^*} |X_n - X| \, dP = 0$, *and*
3. $X_n = E[X | \mathcal{B}_n^*]$ $P - a.e.$.

What if for some $N \geq 1$, $\epsilon_{\omega_n} = 0$ for all $n > N$?

Definition

In this case we say f has **finite Martingale**.

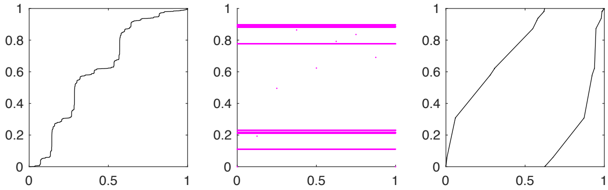


Figure: $C = .1$, $\tau = .9$. $\epsilon_{\omega_n} = 0$ for all $n > 4$.

Theorem

Then f is piecewise linear.

Theorem

f has finite Martingale $\iff$ its Martingale converges to a simple function (i.e. locally constant).

Symmetric rigidity

Theorem (J)

Suppose f and g are both circle endomorphisms of the same topological degree $d \geq 2$ such that each has bounded geometry and preserves the Lebesgue probability measure λ on $\mathbb{T}$.

Suppose h is a symmetric conjugacy between f and g.

- *If $f = q_d$ then h must be the identity.*

Theorem

- *If f has a Martingale with a locally constant limit then h must be the identity.*

Outline of symmetric rigidity proof (1/2)

- symmetrically conjugate $\Rightarrow$ same limits of Martingales

$$X_f(\omega^*) = X_g(\omega^*), \quad P - a.e.$$

- Both limits locally constant $\Rightarrow$ $(X_{f,n})$ and $(X_{g,n})$ must be finite of length n_0

$$X_{n,f}(\omega^*) = X_{n,g}(\omega^*) = c(\omega^*_{n_0}) \quad \forall n \geq n_0.$$

- This implies

$$\frac{|I_{w_{n_0+m-1},f}|}{|I_{\omega_{n_0+m},f}|} = \frac{|F(I_{w_{n_0+m-1},f})|}{|F(I_{\omega_{n_0+m},f})|} \quad \forall m \geq 1,$$

$$\Rightarrow \frac{|F(I_{w_{n_0+m},f})|}{|I_{\omega_{n_0+m},f}|} = \frac{|F(I_{w_{n_0+m-1},f})|}{|I_{\omega_{n_0+m-1},f}|} = \ldots = \frac{|F(I_{w_{n_0},f})|}{|I_{\omega_{n_0},f}|} := \alpha_{w_{n_0}}.$$

- Thus f is piecewise linear

$$F|_{I_{\omega_{n_0}}}(x) = \alpha_{\omega_{n_0}} x + b_{\omega_{n_0}}.$$

Outline of symmetric rigidity proof (2/2)

- Similarly, g is piecewise linear.
- Since $(X_{n,f}) = (X_{n,g})$ for all $n \geq n_0$,

$$\frac{|H(I_{w_{n_0+m-1,f}})|}{|H(I_{w_{n_0+m,f}})|} = \frac{|I_{w_{n_0+m-1,g}}|}{|I_{w_{n_0+m,g}}|} = \frac{|I_{w_{n_0+m-1,f}}|}{|I_{w_{n_0+m,f}}|}.$$

- This implies

$$\frac{|H(I_{w_{n_0+m,f}})|}{|I_{w_{n_0+m,f}}|} = \frac{|H(I_{w_{n_0+m-1,f}})|}{|I_{w_{n_0+m-1,f}}|} = \ldots = \frac{|H(I_{w_{n_0,f}})|}{|I_{w_{n_0,f}}|} := d_{w_{n_0}}$$

- Thus

$$H(x) = d_{w_{n_0}} x + e_{n_0}, \quad \forall x \in I_{w_{n_0}}.$$

- H is symmetric with $H(0) = 0$ and $H(1) = 1 \Rightarrow H(x) = x$.

Thank you very much!