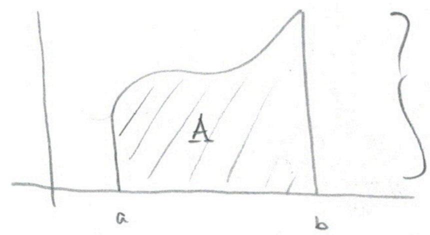
§8.8 IMPROPER INTEGRALS

$$\int_a^b f(x)\,dx =$$



(II) FINITE HEIGHT

f IS BOUNDED ON (a, b)

(I) FINITE WIDTH

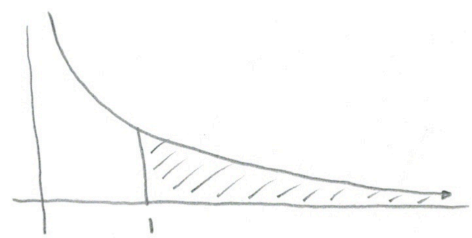
(a, b BOTH FINITE)

"PROPER INTEGRALS"

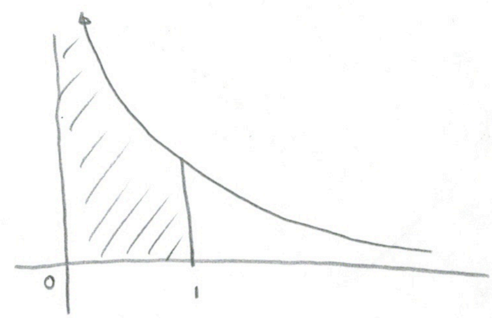
$\therefore$ AREA UNDER CURVE IN THIS CASE IS WELL-DEFINED.

BUT WHAT ABOUT

(I) INFINITE WIDTH (IMPROPER TYPE I) e.g. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$



(II) INFINITE HEIGHT (IMPROPER TYPE II) e.g. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$



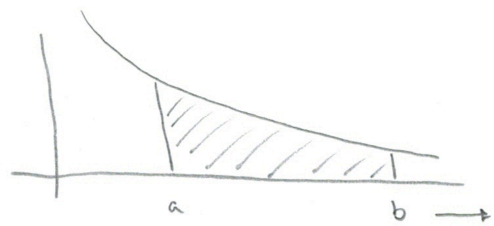
IMPROPER TYPE I:

IF $f(x)$ IS CONTINUOUS ON $[a, \infty)$ THEN

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx = \lim_{b\to\infty} F(b) - F(a)$$

Def: IF THIS LIMIT EXISTS, WE SAY THE IMPROPER INTEGRAL CONVERGES. OTHERWISE, IT DIVERGES. (COULD BE $\pm\infty$ OR COULD OSCILLATE)

$$= \lim_{b\to\infty}$$



SIMILARLY, $\cdot$) IF $f(x)$ IS CONTINUOUS ON $(-\infty, b)$ THEN

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx = F(b) - \lim_{a\to-\infty} F(a)$$

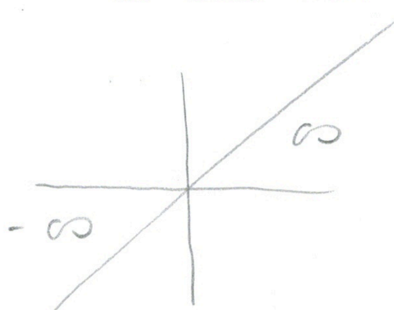
$\cdot$) IF $f(x)$ IS CONTINUOUS ON $(-\infty, \infty)$ THEN

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx$$

$$= \underbrace{\left(F(c) - \lim_{a\to-\infty} F(a)\right)} + \underbrace{\left(\lim_{b\to\infty} F(b) - F(c)\right)}$$

NOTE: IF EITHER OF THESE LIMITS DIVERGES, THE WHOLE THING DIVERGES.

e.g. $\int_{-\infty}^\infty x\,dx$ DIVERGES



$\underline{\underline{\infty - \infty \neq 0}}$

ex. $\int_0^\infty e^{-x}\,dx$, $\int_0^\infty \sin x\,dx$, $\int_0^\infty e^{-x}\sin x\,dx$

$\int_{-\infty}^{\infty} x e^{-x^2}\,dx$ $\quad$ * $\int_1^\infty \frac{1}{x}\,dx$ $\quad$ $\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}}\,dx$

IMPROPER TYPE II :

• IF $f(x)$ IS CONTINUOUS ON $(a, b]$ & DISCONTINUOUS AT a THEN

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx = F(b) - \lim_{t \to a^+} F(t)$$

SIMILARLY • IF $f(x)$ IS CONTINUOUS ON $[a, b)$ & DISCONTINUOUS AT b THEN

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx = \lim_{t \to b^-} F(t) - F(a)$$

• IF $f(x)$ IS CONTINUOUS ON $(a, c) \cup (c, b)$ &

DISCONTINUOUS AT c , THEN

$$\int_a^b f(x)\,dx = \lim_{t \to c^-} \int_a^t f(x)\,dx + \lim_{t \to c^+} \int_t^b f(x)\,dx$$

$$= \underbrace{\lim_{t \to c^-} F(t) - F(a)} + \underbrace{F(b) - \lim_{t \to c^+} F(t)}$$

IF EITHER LIMIT D.N.E. THEN THE WHOLE

INTEGRAL DIVERGES.

ex. $\displaystyle\int_2^4 \frac{1}{(x-2)^{1/3}}\,dx$, $\displaystyle\int_{-3}^{3} \frac{1}{x}\,dx$, $\displaystyle\int_0^1 \ln x\,dx$

$\displaystyle\int_0^1 \frac{4x}{\sqrt{1-x^4}}\,dx$ $\quad * \displaystyle\int_0^1 \frac{1}{x}\,dx$

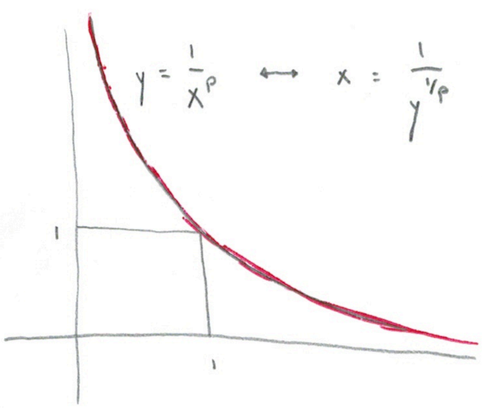
p-TEST TYPE I:

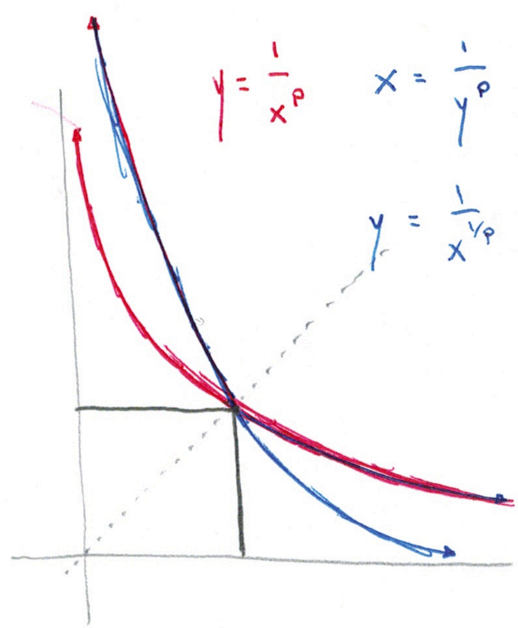
$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \frac{1}{1-p}\, x^{1-p}\Big|_1^t = \frac{1}{1-p}\left[\lim_{t\to\infty} t^{1-p} - 1\right] = \begin{cases} \frac{1}{p-1} & \text{IF } p>1 \\ \infty & \text{IF } p\le 1 \end{cases}$$

$p>0$

$p\neq 1$

p-TEST TYPE II: $\displaystyle\int_0^1 \frac{1}{x^p}\,dx = \Big\{$





$$\int_0^1 \frac{1}{x^p}\,dx = 1 + \int_1^\infty \frac{1}{x^{1/p}}\,dx = \begin{cases} 1 + \frac{1}{\frac{1}{p}-1} & \text{IF } p<1 \quad \left(\frac{1}{1-p} \text{ IF } p<1\right) \\ \infty & \text{IF } p\ge 1 \end{cases}$$

THM: DIRECT COMPARISON THM

LET f, g CONTINUOUS ON $[a, \infty)$ WITH $0 \le f(x) \le g(x)$ FOR ALL $x \ge a$.

1. IF $\int_a^\infty g(x)\,dx$ CONVERGES THEN $\int_a^\infty f(x)\,dx$ CONVERGES

2. IF $\int_a^\infty f(x)\,dx$ DIVERGES THEN $\int_a^\infty g(x)\,dx$ DIVERGES.

e.x. $\int_\pi^\infty \frac{1+\sin x}{x^2}$ CONVERGE OR DIVERGE? $\int_0^1 \frac{\sin^2 x}{\sqrt{x}}$

$\int_\pi^\infty \frac{2+\sin x}{x}\,dx$ $\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$

THM. LIMIT COMPARISON TEST

IF f & g ARE POSITIVE & CONTINUOUS ON $[a, \infty)$ AND IF

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty,$$

THEN $\int_a^x f(x)\,dx$ AND $\int_a^x g(x)\,dx$

EITHER BOTH CONVERGE OR BOTH DIVERGE

ex. $\int_1^\infty \frac{1}{e^x-1}\,dx$ $\int_4^\infty \frac{2\,dt}{t^{3/2}-1}$ $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$