§ 8.4 TRIG SUBSTITUTIONS

GIVEN

$$\int_0^{\pi/2} \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta \qquad \begin{array}{c}\text{USE } x\text{-SUB.}\\ x=\sin\theta \\ dx = \cos\theta\,d\theta\end{array} \qquad \rightsquigarrow \int_0^1 \sqrt{1-x^2}\,dx$$

SAME ANSWER.

DO YOU SEE WHY GEOMETRICALLY?

IN TRIG SUB., WE REPLACE THE RIGHT WITH THE LEFT!

WHY? TRIG IDENTITIES!

$$\to \int_0^{\pi/2} \underbrace{\sqrt{\cos^2\theta}}_{|\cos\theta|}\,\cos\theta\,d\theta = \int_0^{\pi/2} \cos^2\theta\,d\theta = \frac{1}{2}\int_0^{\pi/2} 1+\cos 2\theta\,d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2} + \frac{1}{2}(0-0)\right) = \frac{\pi}{4}$$

(1)

IN GENERAL, EXPRESSION: $\sqrt{a^2-x^2}$, $-a \le x \le a$ (ASSUME $a>0$)

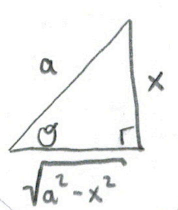
SUBSTITUTION: $x = a\sin\theta$, $-\frac{\pi}{2} \le \theta = \sin^{-1}\frac{x}{a} \le \frac{\pi}{2}$

$dx = a\cos\theta$

SIMPLIFY: $\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2}\sqrt{1-\sin^2\theta}$

$= a\sqrt{\cos^2\theta} = a|\cos\theta| = a\cos\theta \quad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$

$\sin\theta = \frac{x}{a}$



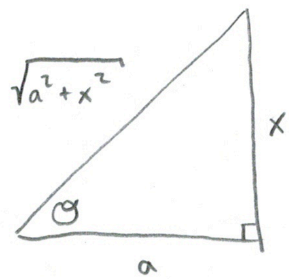
$a\cos\theta = \sqrt{a^2-x^2}$ ✓

(2)

EXPRESSION : $\sqrt{a^2 + x^2}$, $-\infty < x < \infty$, $a > 0$

SUBSTITUTE : $x = a \tan\theta \longrightarrow \tan\theta = \frac{x}{a}$, $\theta = \tan^{-1}\frac{x}{a}$

$dx = a\sec^2\theta$ $\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$



$a\sec\theta = \sqrt{a^2 + x^2}$

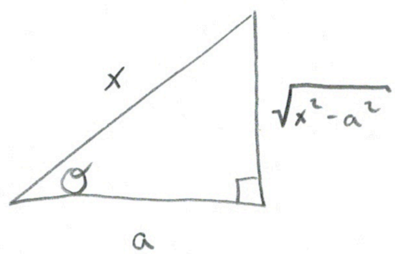
(3)

EXPRESSION : $\sqrt{x^2 - a^2}$, $x \le -a$ or $x \ge a$, $a > 0$

SUBSTITUTE : $x = a\sec\theta \longrightarrow \sec\theta = \frac{x}{a}$, $\theta = \sec^{-1}\frac{x}{a}$

$0 \le \theta < \frac{\pi}{2}$ IF $\frac{x}{a} \ge 1$

$\frac{\pi}{2} < \theta \le \pi$ IF $\frac{x}{a} \le -1$

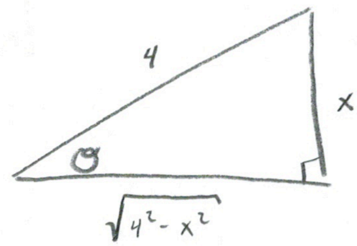


$a\tan\theta = \sqrt{x^2 - a^2}$

ex. (EASY) $\displaystyle\int \frac{1}{\sqrt{16 - x^2}}\, dx \quad \left(= \sin^{-1}\left(\frac{x}{4}\right) + C \right)$

$x = 4 \sin\theta \quad , \quad \sin\theta = \frac{x}{4}$

$\underline{dx = 4\cos\theta \, d\theta}$



$\underline{4\cos\theta = \sqrt{4^2 - x^2}}$

$\rightsquigarrow \displaystyle\int \frac{4\cos\theta}{4\cos\theta}\, d\theta = \theta + C$

$= \sin^{-1}\frac{x}{4} + C$ ✓

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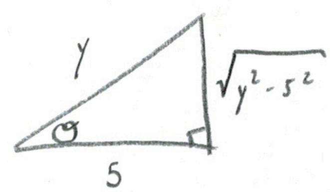
ex. $\displaystyle\int \frac{\sqrt{y^2 - 25}}{y^3}\, dy \quad , \quad y > 5$



$= 5\tan\theta$

Let $y = 5\sec\theta \longrightarrow \sec\theta = \frac{y}{5}$

$dy = 5\sec\theta\tan\theta\, d\theta$

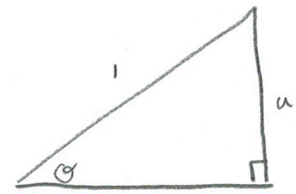
$\rightsquigarrow \displaystyle\int \frac{5\tan\theta}{(5\sec\theta)^3} \cdot 5\sec\theta\tan\theta\, d\theta = \frac{1}{5}\int \frac{\tan^2\theta}{\sec^2\theta}\, d\theta$

$= \displaystyle\frac{1}{5}\int \sin^2\theta\, d\theta = \frac{1}{10}\int 1 - \cos 2\theta\, d\theta = \frac{1}{10}\left[\theta - \underbrace{\frac{1}{2}\sin 2\theta}_{\sin\theta\cos\theta}\right] + C$

$= \displaystyle\frac{1}{10}\left[\theta - \sin\theta\cos\theta\right] + C \rightsquigarrow \boxed{\frac{1}{10}\left(\sec^{-1}\frac{y}{5} - \frac{\sqrt{y^2 - 5^2}}{y} \cdot \frac{5}{y}\right) + C}$

ex. $\displaystyle\int \frac{\sqrt{1-(\ln x)^2}}{x \ln x}\,dx$ $\qquad u = \ln x$

$du = \frac{1}{x}\,dx$



$\displaystyle\rightsquigarrow \int \frac{\sqrt{1-u^2}}{u}\,du$ $\qquad$ LET $u = \sin\theta$

$du = \cos\theta$

$\sqrt{1-u^2} = \cos\theta$

$$\rightsquigarrow \int \frac{\cos\theta}{\sin\theta}\cos\theta\,d\theta = \int \frac{\cos^2 x}{\sin x}\,dx = \int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta$$

$$= \int \csc\theta - \sin\theta\,d\theta = -\ln\left|\csc x + \cot x\right| + \cos\theta + C$$

ex. $\displaystyle\int_0^1 \frac{1}{(4-x^2)^{3/2}}\,dx = \int_0^1 \frac{1}{\sqrt{4-x^2}^{\,3}}\,dx$

$\sqrt{4-x^2} = 2\cos\theta$

Let $x = 2\sin\theta \rightarrow \sin\theta = \frac{x}{2}$

$dx = 2\cos\theta\,d\theta$

$\theta = \sin^{-1}\left(\frac{x}{2}\right)$

$$\rightsquigarrow \int_0^{\pi/6} \frac{1}{(2\cos\theta)^3}\,2\cos\theta\,d\theta = \frac{1}{4}\int \sec^2\theta\,d\theta$$

$$= \frac{1}{4}\tan\theta\Big|_0^{\pi/6} = \frac{1}{4}\left(\frac{1}{\sqrt{3}} - 0\right) = \boxed{\frac{\sqrt{3}}{12}}$$