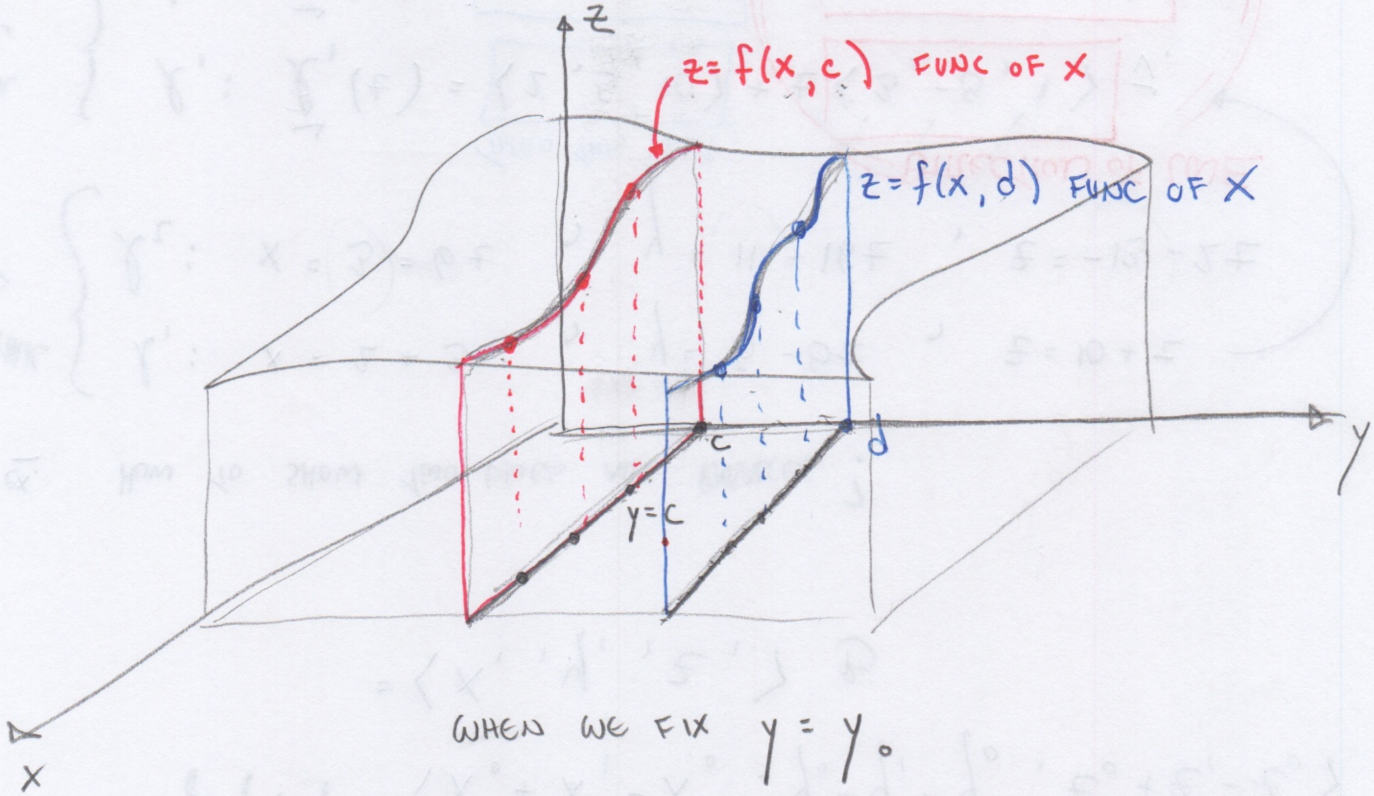
§ 14.3 PARTIAL DERIVATIVES.

$z = f(x, y)$ GRAPH/ SURFACE



WHEN WE FIX $y = y_0$

$z = f(x, y_0)$ IS NOW A FUNC OF 1 VAR, X.

NOT A VARIABLE (CONSTANT)

WE GET A DIFF. FUNC. OF X

FOR EVERY VALUE OF y.

EVERY TIME YOU CHANGE THE VALUE y,

YOU GET A NEW FUNC. OF x.

ex. SUPPOSE $f(x,y) = 3x^4y^5 - 5xy + \dfrac{x}{y}$

IF WE FIX $y = 2$: $f(x,2) = 3x^4(2)^5 - 5x(2) + \dfrac{x}{2}$

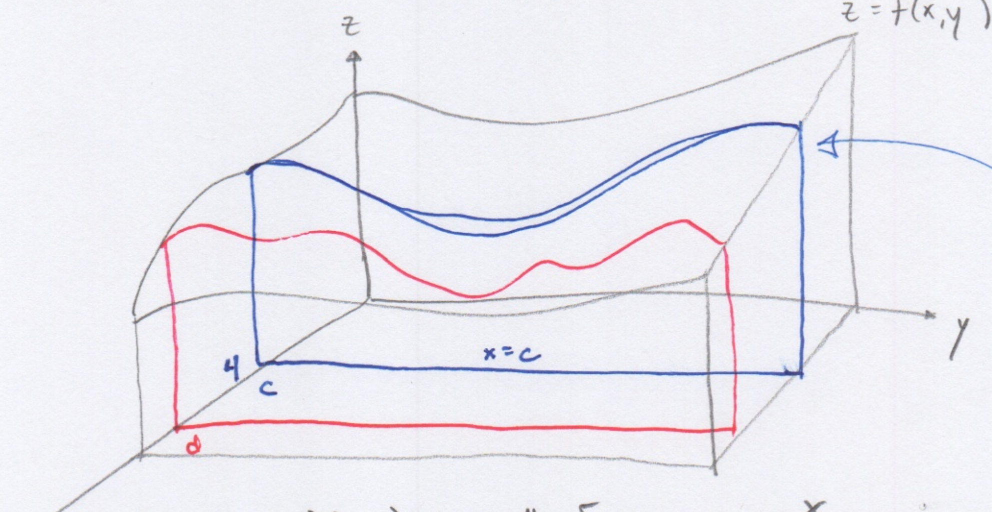
$$= \underbrace{\boxed{96x^4 - 10x + \frac{x}{2}}}_{\text{FUNC. OF } x \ (y=2)} \ y=2$$

IF WE FIX $y = -1$: $f(x,-1) = \boxed{-3x^4 + 5x - x} \ y=-1$

SO WE CAN THINK OF $f(x,y)$ AS A FAMILY OF FUNCTIONS OF x : ONE FOR EVERY VALUE OF y.

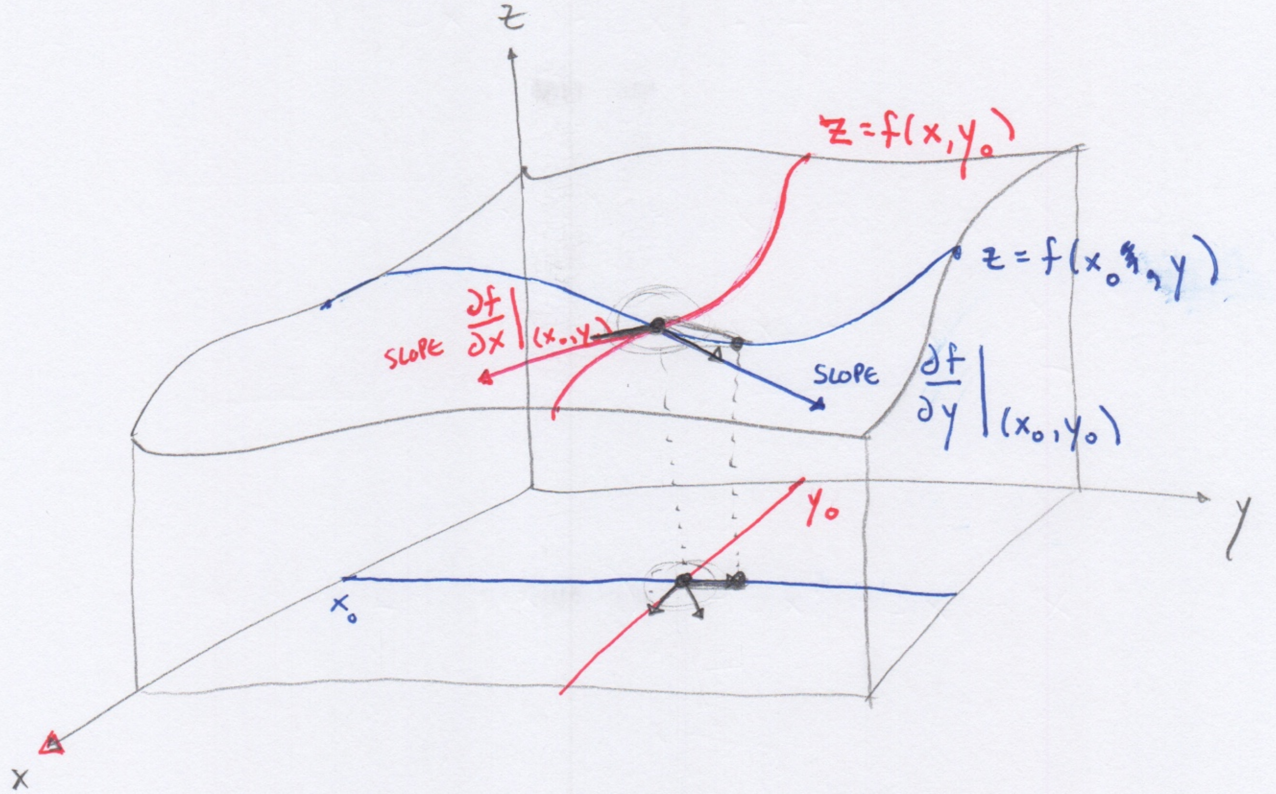
SIMILARLY, WE CAN FIX $x = c$ AND CONSIDER

$f(x,y) \longrightarrow f(c,y)$ 1 VAR.



ex. $f(x,y) = 3x^4y^5 - 5xy + \dfrac{x}{y}$

FIX $x = 4$: $f(4,y) = 768y^5 - 20y + \dfrac{4}{y}$



DERIVATIVE IS RATE OF CHANGE. $\dfrac{\Delta \text{ OUTPUT}}{\Delta \text{ INPUT}}$ → 2 INPUTS!

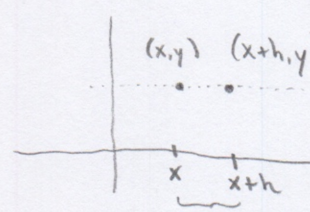
W.R.T. x / W.R.T. y

PARTIAL DERIVATIVES

THE PARTIAL DERIVATIVE OF $f(x,y)$ W.R.T. x IS

"PARTIAL f PARTIAL x"

$$\frac{\partial f}{\partial x} = f_x = \lim_{h \to 0} \underbrace{\frac{f(x+h, y) - f(x, y)}{h}}_{\text{A FUNCTION OF } x \text{ \& } y}$$



& THE PARTIAL DERIVATIVE OF $f(x,y)$ WRT x AT (x_0, y_0) IS

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = f_x(x_0, y_0) = \text{PLUG IN } x = x_0,\ y = y_0 \text{ INTO}$$

THE PARTIAL DERIVATIVE OF $f(x,y)$ WRT. y IS

$$\frac{\partial f}{\partial y} = f_y = \lim_{h \to 0} \underbrace{\frac{f(x, y+h) - f(x,y)}{h}}_{\text{FUNCTION OF } x \text{ \& } y}$$

$\frac{\Delta \text{ OUTPUT}}{\Delta \text{ INPUT } y}$

& THEN PARTIAL DERIVATIVE OF $f(x,y)$ WRT y AT (x_0, y_0) IS

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = f_y(x_0, y_0) = \text{PLUG IN } \begin{array}{l} x = x_0 \\ y = y_0 \end{array}$$

INTO

$Z = X$

EVALUATING PARTIAL DERIVATIVES:

$\frac{\partial}{\partial x_y} f(x,y) \longrightarrow$ TREAT ALL VARIABLES EXCEPT x y LIKE A CONSTANT & TAKE ORDINARY DERIVATIVE WRT. x y

PARTIAL DERIVATIVE WRT. x y

$$\frac{\partial}{\partial x} f(x, \overbrace{y, z}^{\text{CONST.}})$$

CONST.

$\underline{\text{ex.}}$ $f(x,y) = 8x^2y^3 + 5x^{-2}y^{-4} + \sin(xy)$

$$\frac{\partial f}{\partial x} = f_x = \frac{\partial}{\partial x}\left[\underbrace{(8y^3)}_{\text{COEFF.}}x^2 + \underbrace{(5y^{-4})}_{\text{COEFF.}}x^{-2} + \sin(\underbrace{(y)}_{\text{COEFF.}}x)\right]$$

$$= 8y^3 \cdot 2x + 5y^{-4} \cdot {}^{-}2x^{-3} + y\cos(yx)$$

$$= \boxed{16y^3 - 10y^{-4}x^{-3} + y\cos(yx)}$$

ex. $f(x,y) = \ln\left(\sqrt{x^2+y^2}\right) = \ln\left[(x^2+y^2)^{1/2}\right] = \frac{1}{2}\ln(x^2+y^2)$

FIND $\left.\dfrac{\partial f}{\partial x}\right|_{(2,3)}$

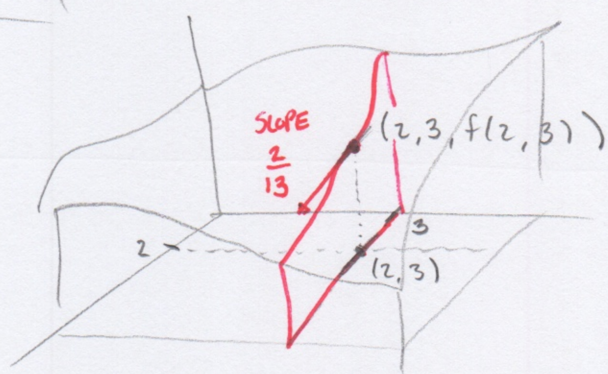
① FIND $\dfrac{\partial f}{\partial x}$

② PLUG IN $(2,3)$

① $$\frac{\partial f}{\partial x} = f_x = \left(\frac{1}{\cancel{2}}\right)\cdot\left(\frac{1}{x^2+y^2}\right)\cdot\left(\cancel{2}x + 0\right) = \frac{x}{x^2+y^2}$$

② $$\frac{2}{2^2+3^2} = \boxed{\frac{2}{13}}$$

WHAT IS THIS?



CALC I

$$\left(\frac{df}{dx} = \frac{dy}{dx}\right)$$

$$z = f(x,y)$$

CALC II

$$\left\{\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial z}{\partial x} \\ \frac{\partial f}{\partial y} &= \frac{\partial z}{\partial y} \end{aligned}\right.$$

2^{nd} DERIVATIVES

$$f(x,y) = x^2 + y^4 + x^3y^5$$

2

(1^{st} DERIVATIVES)

$$\left(\frac{\partial f}{\partial x}\right) = 2x + 3x^2y^5$$

f_x

$$\left(\frac{\partial f}{\partial y}\right) = 4y^3 + 5x^3y^4$$

f_y

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$= f_{xx}$$

$$= 2 + 6xy^5$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x}$$

$$= f_{xy}$$

$$= 15x^2y^4$$

$$\frac{\partial f}{\partial y} = 4y^3 + 5x^3y^4$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y}$$

$$= f_{yx}$$

$$= 15x^2y^4$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$= f_{yy}$$

$$= 12y^2 + 20x^3y^3$$

SUMMARY:

2 1^{st} DERIV. f_x , f_y

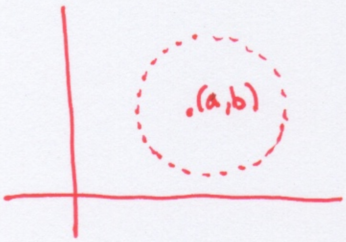
4 2^{nd} DERIV. $f_{xx}, \underbrace{f_{xy}, f_{yx}}_{\text{MIXED } 2^{nd} \text{ DERIVATIVES}}, f_{yy}$

The Mixed Derivative Theorem

You may have noticed that the "mixed" second-order partial derivatives

$$\frac{\partial^2 f}{\partial y\, \partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial x\, \partial y}$$

in Example 9 are equal. This is not a coincidence. They must be equal whenever f, f_x, f_y, f_{xy}, and f_{yx} are continuous, as stated in the following theorem. However, the mixed derivatives can be different when the continuity conditions are not satisfied (see Exercise 82).



THEOREM 2—The Mixed Derivative Theorem
If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$\boxed{f_{xy}(a, b) = f_{yx}(a, b).}$$

HISTORICAL BIOGRAPHY
Alexis Clairaut
(1713–1765)
www.goo.gl/MYhzm3

Theorem 2 is also known as Clairaut's Theorem, named after the French mathematician Alexis Clairaut, who discovered it. A proof is given in Appendix 9. Theorem 2 says that to calculate a mixed second-order derivative, we may differentiate in either order, provided the continuity conditions are satisfied. This ability to proceed in different order sometimes simplifies our calculations.

EXAMPLE 10 Find $\dfrac{\partial^2 w}{\partial x\, \partial y}$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$

Solution The symbol $\partial^2 w/\partial x\, \partial y$ tells us to differentiate first with respect to y and then with respect to x. However, if we interchange the order of differentiation and differentiate first with respect to x we get the answer more quickly. In two steps,

$$\frac{\partial w}{\partial x} = y \quad \text{and} \quad \frac{\partial^2 w}{\partial y\, \partial x} = 1.$$

If we differentiate first with respect to y, we obtain $\partial^2 w/\partial x\, \partial y = 1$ as well. We can differentiate in either order because the conditions of Theorem 2 hold for w at all points (x_0, y_0). ■

Partial Derivatives of Still Higher Order

Although we will deal mostly with first- and second-order partial derivatives, because these appear the most frequently in applications, there is no theoretical limit to how many times we can differentiate a function as long as the derivatives involved exist. Thus, we get third- and fourth-order derivatives denoted by symbols like

$$\frac{\partial^3 f}{\partial x\, \partial y^2} = f_{yyx},$$

$$\frac{\partial^4 f}{\partial x^2\, \partial y^2} = f_{yyxx},$$

and so on. As with second-order derivatives, the order of differentiation is immaterial as long as all the derivatives through the order in question are continuous.

ex. $w = \dfrac{x-y}{x^2+y}$ CALCULATE ALL 2^{nd}-ORDER PARTIAL DERIV.'S

$$x^2+y \quad -2x^2 \quad -2xy$$

$$\frac{\partial w}{\partial x} = \frac{(x^2+y)1-(x-y)\cdot 2x}{(x^2+y)^2} = \frac{-x^2-2xy+y}{(x^2+y)^2}$$

1. $$\frac{\partial^2 w}{\partial x^2} = \frac{(x^2+y)^2(-2x-2y)-(-x^2-2xy+y)2(x^2+y)2x}{(x^2+y)^4}$$

2. $$\frac{\partial^2 w}{\partial y\,\partial x} = \frac{(x^2+y)^2(-2x+1)-(-x^2-2xy+y)2(x^2+y)}{(x^2+y)^4}$$

$$-x^2 \quad \cancel{-y} \quad -x \quad \cancel{+y}$$

$$\frac{\partial w}{\partial y} = \frac{(x^2+y)(-1)-(x-y)}{(x^2+y)^2} = \frac{-x^2-x}{(x^2+y)^2}$$

SAME!

3. $$\frac{\partial^2 w}{\partial x\,\partial y} = \frac{(x^2+y)^2(-2x-1)-(-x^2-x)2(x^2+y)2x}{(x^2+y)^4}$$

4. $$\frac{\partial^2 w}{\partial y^2} = \frac{(x^2+y)^2\,0-(-x^2-x)2(x^2+y)(0+1)}{(x^2+y)^4}$$

ex. $g(u,v) = v^2 \underbrace{e^{2u/v}}$

$$g_u = \underbrace{v^2\left[e^{\frac{2}{v}u} \cdot \frac{2}{v}\right]} = 2v\,e^{2u/v}$$

$$\frac{\partial}{\partial u}\left[\underbrace{v^2}_{\text{COEFF.}}\, e^{\underbrace{\frac{2}{v}}_{\text{COEFF.}}u}\right]$$

$$g_v = \frac{\partial}{\partial v}\left[(v^2)\left(e^{2uv^{-1}}\right)\right]$$

$$= 2v\left(e^{2uv^{-1}}\right) + \cancel{v^2}\left(e^{2uv^{-1}} \cdot \left(-2u\,\cancel{v^{-2}}\right)\right)$$

$$= 2v\,e^{2uv^{-1}} - 2u\,e^{2uv^{-1}}$$