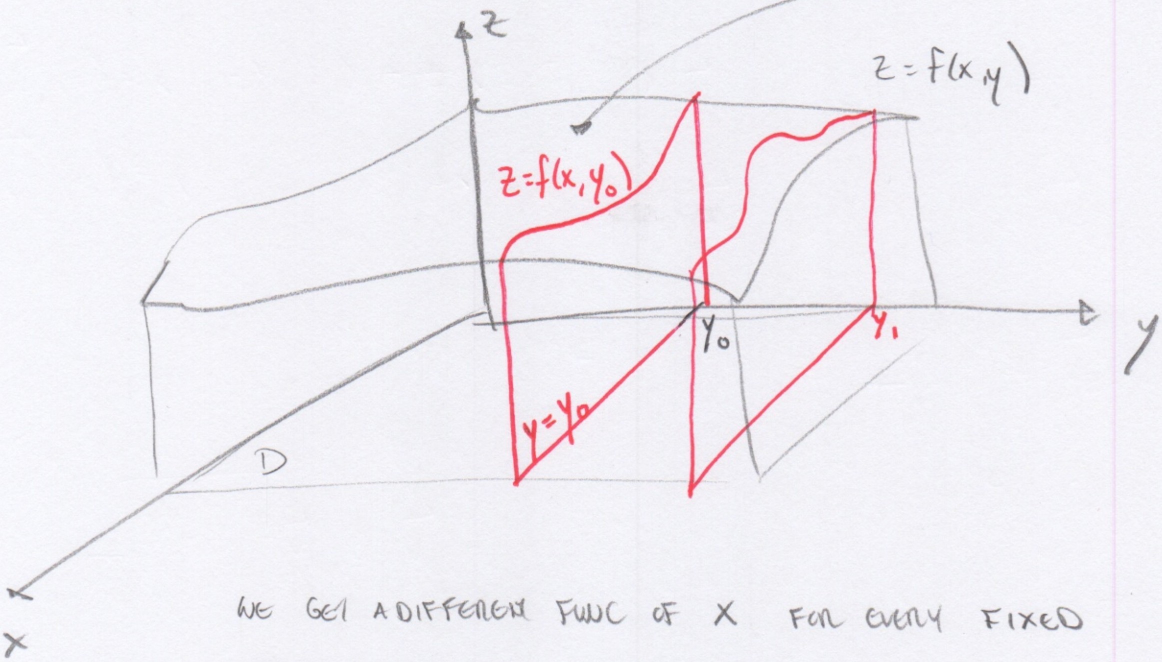
§14.3 PARTIAL DERIVATIVES

GIVEN $f(x,y)$ WITH DOMAIN D, IF WE FIX $y = y_0$, THEN $f(x, y_0)$ IS REALLY A FUNCTION OF X. (1-VAR)



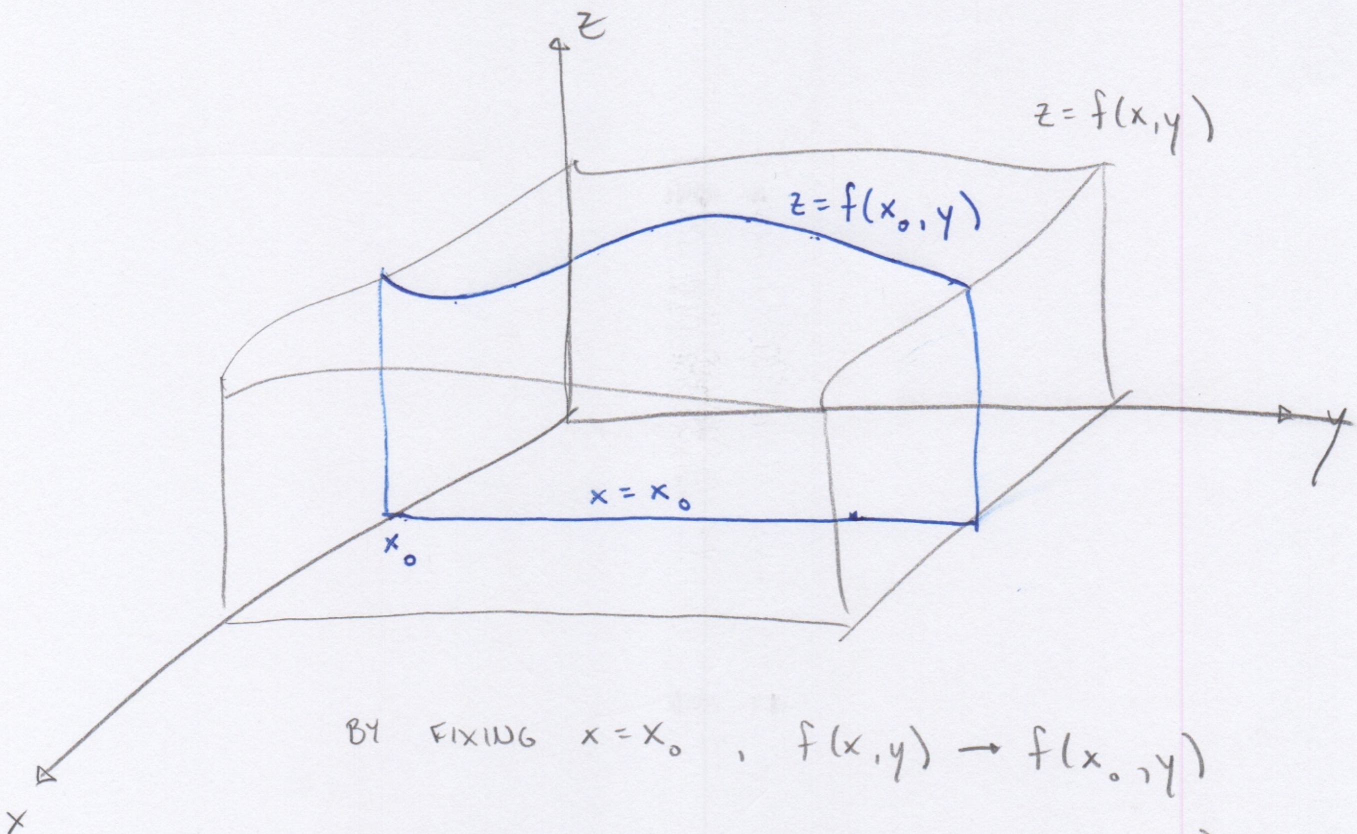
WE GET A DIFFERENT FUNC OF X FOR EVERY FIXED VALUE OF y.

e.g. LET $f(x,y) = 2xy^2 + 3y + 2\sin(xy)$

FIX $y = 3$. NOW $f(x,y)\Big|_{y=3} = f(x,3)$

$$= 2x(3)^2 + 3(3) + 2\sin(x(3))$$

$$\begin{cases} f(x,3) = 18x + 9 + 2\sin(3x) \\ f(x,-2) = 8x - 6 + 2\sin(-2x) \end{cases}$$



BY FIXING $x = x_0$, $f(x,y) \to f(x_0, y)$

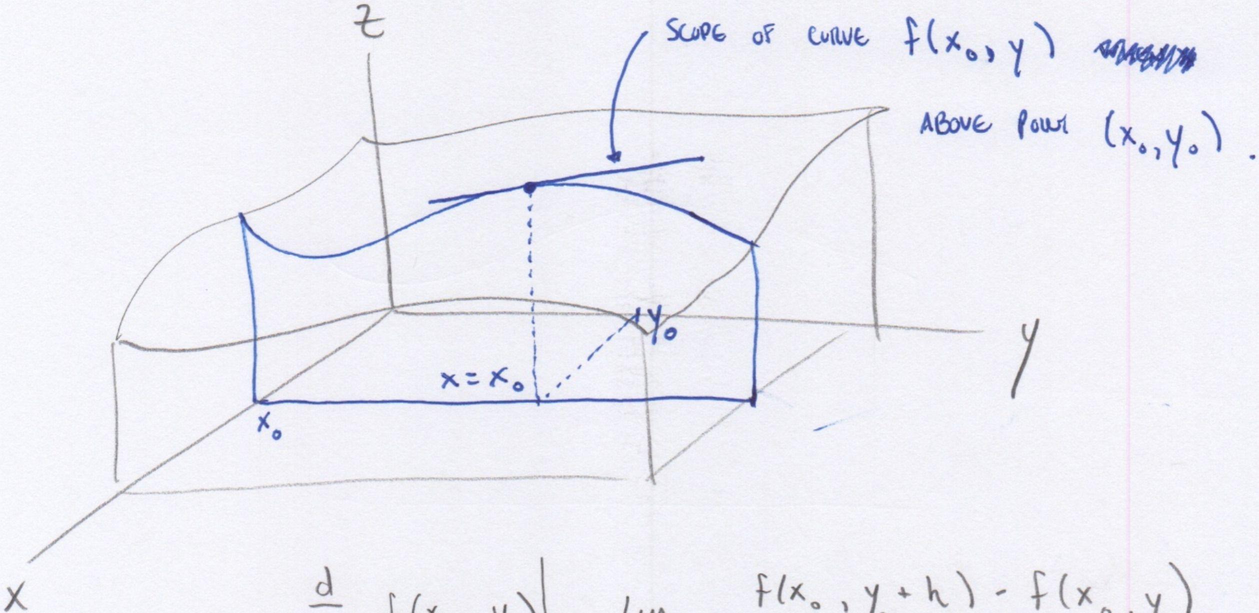
BECOMES A FUNCTION OF y (1-VAR.).

DIFFERENT VALUES FOR x GIVE DIFFERENT FUNCTIONS OF y.

e.g. $f(x,y) = x^y + y^x - \ln(x^2+y^2)$

FIX $x = 5$: $f(5,y) = 5^y + y^5 - \ln(25+y^2)$

FIX $x = 3$: $f(3,y) = 3^y + y^3 - \ln(9+y^2)$



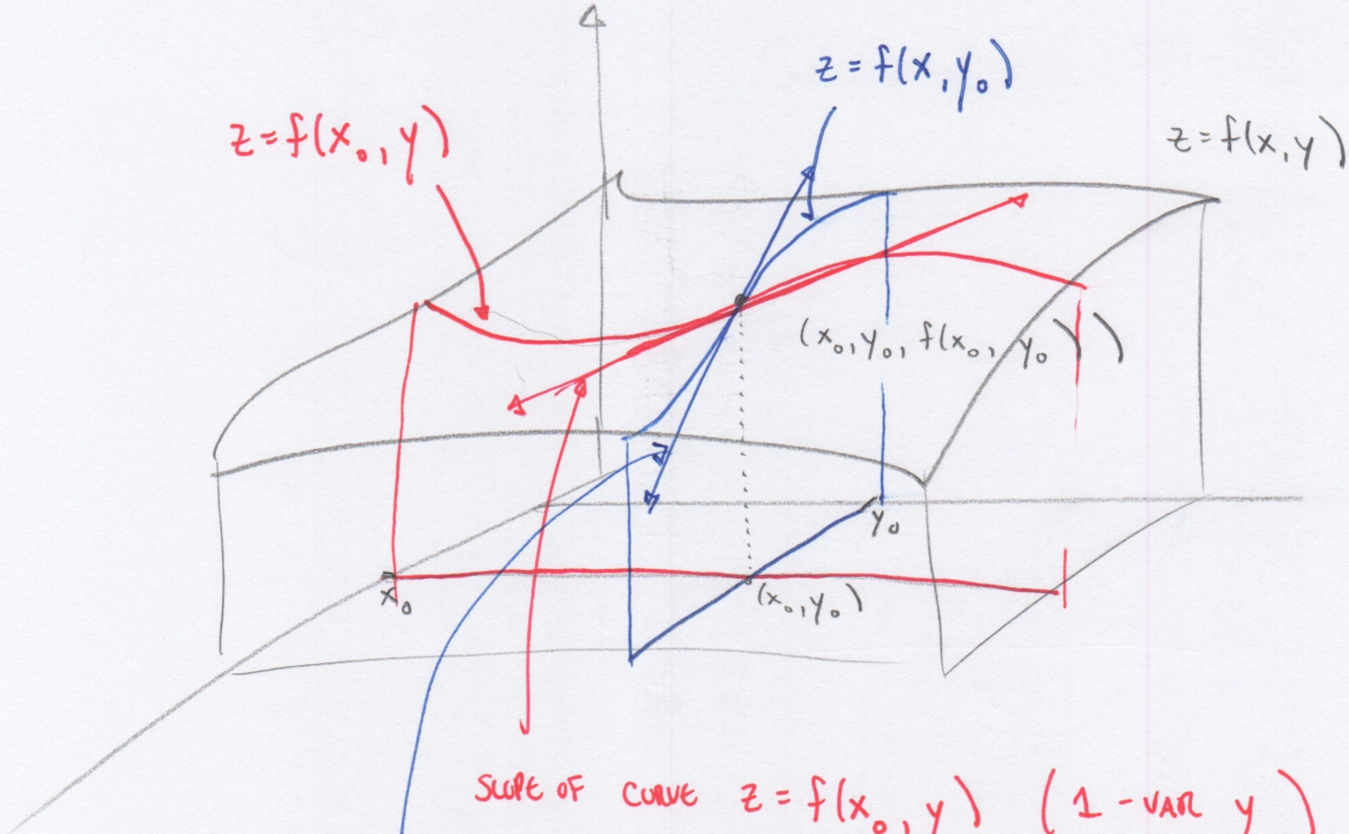
$$\frac{d}{dy} f(x_0, y)\bigg|_{(x_0,y_0)} = \lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

Def: THE PARTIAL DERIVATIVE OF $f(x,y)$ WITH RESPECT TO x, EVALUATED AT THE POINT (x_0, y_0) IS

$$\frac{\partial}{\partial x}\bigg|_{(x_0,y_0)} = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

AND THE PARTIAL DERIVATIVE OF $f(x,y)$ W.R.T. y, EVALUATED AT THE POINT (x_0, y_0) IS

$$\frac{\partial}{\partial y}\bigg|_{(x_0,y_0)} = \lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$



SLOPE OF CURVE $z=f(x_0,y)$ (1-VAR y)

ABOVE (x_0,y_0) IS $\left.\dfrac{\partial f}{\partial y}\right|_{(x_0,y_0)}$

SLOPE OF CURVE $z=f(x,y_0)$ (1-VAR x)

ABOVE (x_0,y_0) IS $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)}$

RATES OF CHANGE:

$\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ IS THE RATE AT WHICH z CHANGES WITH RESPECT TO x WHEN $x=x_0$, $y=y_0$.

$\left.\dfrac{\partial f}{\partial y}\right|_{(x_0,y_0)}$ . . . .

. . y . . . .

e.g. LET $f(x,y) = 3xy + 4x^3y^5 - x\sin(y)$

FIND $\left.\dfrac{\partial f}{\partial x}\right|_{(1,2)}$ FIX $y=2$

BY DEF: $\displaystyle\lim_{h\to 0} \frac{f(1+h,2)-f(1,2)}{h}$

$= 3(1+h)2 + 4(1+h)^3 2^5 - x\sin(2) - 134$

HERE'S WHAT YOU DO: TREAT y AS A CONSTANT.

$$\frac{\partial}{\partial x}\left[3xy + 4x^3y^5 - x\sin(y)\right]$$

$$\frac{\partial f}{\partial x} = 3y + 4y^5(3x^2) - \sin(y)$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,2)} = \boxed{3(2) + 4(2)^5 \cdot 3(1)^2 - \sin(2)}$$

$=$

TO EVALUATE THE PARTIAL DERIVATIVE OF f W.R.T. x $\frac{\partial f}{\partial x}$, TREAT y AS A CONSTANT & USE NORMAL RULES FOR DERIVATIVES IN 1-VAR (x). TO EVALUATE $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ DO THE SAME THING & THEN PLUG IN $x = x_0$, $y = y_0$.

ex. GIVEN $f(x,y) = (2x-3y)^5$.

•) $\boxed{\frac{\partial f}{\partial x} = 5(2x-3y)^4 \cdot 2}$

y IS CONSTANT

NOTE $\frac{\partial}{\partial x} g(y) = 0$

•) $\boxed{\frac{\partial f}{\partial y} = 5(2x-3y)^4 \cdot (-3)}$

x IS CONSTANT.

ex. $f(x,y) = e^{xy} \ln y$ FIND $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$

• $\boxed{\frac{\partial f}{\partial x} = \ln y \cdot e^{xy} \cdot y}$

• $\frac{\partial f}{\partial y} = \left(\frac{\partial}{\partial y} e^{xy}\right) \cdot \ln y + e^{xy} \cdot \left(\frac{\partial}{\partial y} \ln y\right)$

$= \boxed{x e^{xy} \cdot \ln y + e^{xy} \cdot \frac{1}{y}}$

FUNCTIONS OF 3 OR MORE VARIABLES:

$\frac{\partial}{\partial x} f(x,y,z)$ TREAT ALL VARIABLES <u>OTHER THAN</u> x AS CONSTANTS & TAKE THE DERIVATIVE W.R.T. x.

$\frac{\partial}{\partial y} f(x,y,z)$ .... OTHER THAN y ...
.... y.

$\frac{\partial}{\partial z} f(x,y,z)$ .... OTHER THAN z ...
.... z.

<u>ex.</u> $f(x,y,z) = xy + yz + xz$

$$\frac{\partial f}{\partial x} = y + z, \quad \frac{\partial f}{\partial y} = x + z \cancel{+0}$$

$$\frac{\partial f}{\partial z} = 0 + y + x. \qquad f_z(-1,2,-4) = (2) + (-1) = \boxed{1}$$

NOTATION: $\frac{\partial f}{\partial x} = f_x, \quad \left.\frac{\partial f}{\partial x}\right|_{(x_0,y)} = f_x(x_0,y_0)$

$$\frac{\partial f}{\partial y} = f_y, \quad \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = f_y(x_0,y_0)$$

HIGHER ORDER PARTIAL DERIVATIVES :

GIVEN $f(x,y)$, IT HAS 2 PARTIAL DERIVATIVES

$$\frac{\partial f}{\partial x} = f_x(x,y) \quad \& \quad \frac{\partial f}{\partial y} = f_y(x,y)$$

EACH OF THESE HAS 2 PARTIAL DERIVATIVES!

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}(x,y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}(x,y)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}(x,y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}(x,y)$$

GIVEN $f(x,y)$, THERE ARE 4 2nd DERIVATIVES :

$$\frac{\partial f}{\partial x^2} = f_{xx} \; , \; \underbrace{\frac{\partial^2 f}{\partial x \partial y} = f_{yx} \; , \; \frac{\partial^2 f}{\partial y \partial x} = f_{xy}} \; , \; \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

MIXED DERIVATIVE THEOREM:

IF $f(x,y)$ & ITS PARTIAL DERIVATIVES f_x, f_y, f_{xy}, f_{yx} ARE DEFINED THROUGHOUT AN OPEN REGION CONTAINING A POINT (a,b) AND ARE ALL CONTINUOUS AT (a,b), THEN

$$\underbrace{f_{xy}(a,b) = f_{yx}(a,b)}_{\text{MIXED PARTIALS ARE THE SAME!}}$$

ex. $f(x,y) = \sin(xy)$. FIND ALL 2^{nd} DERIVATIVES.

$$f_x = y\cos(xy) \qquad f_y = x\cos(xy)$$

$$f_{xx} = \frac{\partial}{\partial x}\left[f_x\right] = -y^2\sin(xy)$$

$$f_{xy} = \frac{\partial}{\partial y}\left[f_x\right] = 1\cdot\cos(xy) + y\cdot\left(-x\sin(xy)\right)$$

$$= \cos(xy) - xy\sin(xy)$$

SAME!

$$f_{yy} = \frac{\partial}{\partial y}\left[f_y\right] = -x^2\sin(xy)$$

$$f_{yx} = \frac{\partial}{\partial x}\left[f_y\right] = 1\cdot\cos(xy) + x\left(-y\sin(xy)\right)$$

$$= \cos(xy) - xy\sin(xy)$$