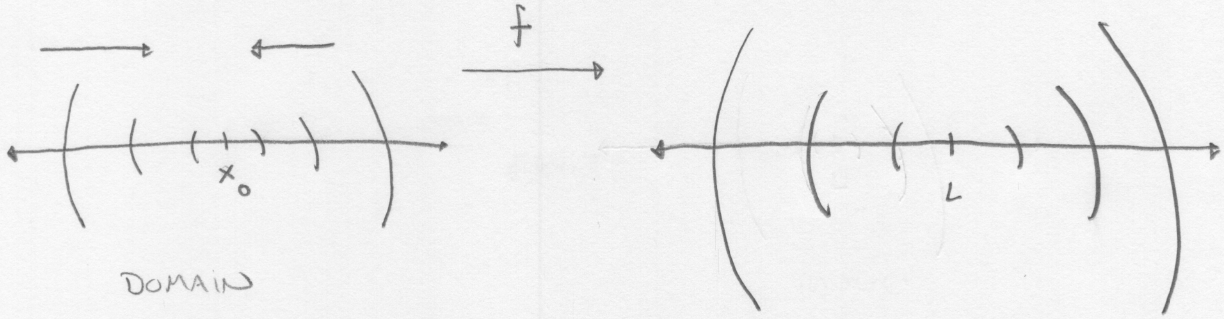
§14.2 Limits & Continuity in Higher Dimensions

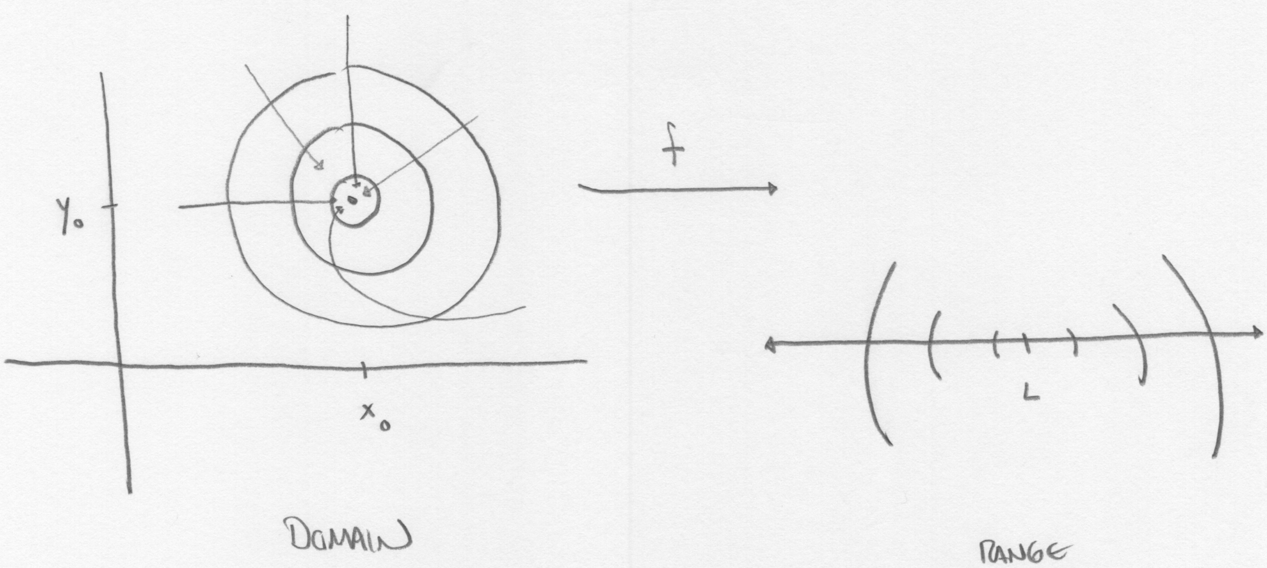
When inputs get closer & closer to a point, outputs gets closer & closer to a limit.

1 var: $\lim\limits_{x \to x_0} f(x) = L$



- Two ways to approach x_0: $x \to x_0^-$, $x \to x_0^+$.

2 var: $\lim\limits_{(x,y) \to (x_0,y_0)} f(x,y) = L$



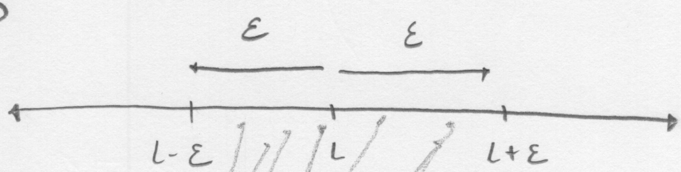
- ∞ ways to approach (x_0, y_0): along hor/vert lines, other line, and even along curves!

Def: $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ MEANS

$\forall \varepsilon > 0$, $\exists \delta > 0$ SUCH THAT

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \varepsilon \Rightarrow |f(x,y) - L| < \varepsilon .$$

THAT IS, GIVEN $\varepsilon > 0$



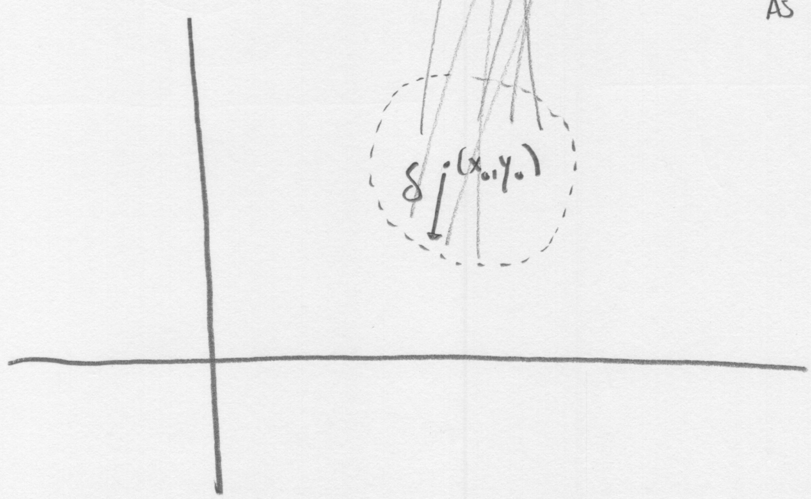
THERE IS A DISK OF RADIUS $\delta > 0$ AROUND (x_0, y_0)

S.T. FOR ALL (x,y) IN THIS DISK & IN DOMAIN, $(x,y) \neq (x_0,y_0)$,

$f(x,y) \in (L-\varepsilon, L+\varepsilon)$

AS $\varepsilon \to 0$,

$\delta \to 0$.



PROPERTIES OF LIMITS OF FUNC. OF 2 VAR.'S

ASSUME $\underbrace{\lim_{(x,y)\to(x_0,y_0)} f(x,y)}_{\lim f}$ & $\underbrace{\lim_{(x,y)\to(x_0,y_0)} g(x,y)}_{\lim g}$ EXIST

THEN

i. $\lim(f \pm g) = \lim f \pm \lim g$

ii. $\lim(kf) = k \lim f$

iii. $\lim(fg) = (\lim f)(\lim g)$

$\lim\left(\frac{f}{g}\right) = \frac{\lim f}{\lim g}$ ← AS LONG AS THIS IS NOT ZERO.

iv. $\lim(f^n) = (\lim f)^n$

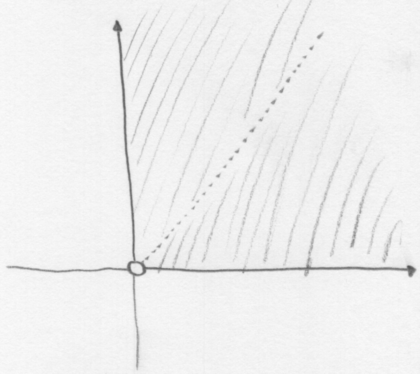
$\lim\left(\sqrt[n]{f}\right) = \sqrt[n]{\lim f}$

* SAME PROPERTIES AS LIMITS IN 1 VARIABLE

ex. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{\sqrt{xy+3}-1}{x^2+y^2}$ PLUG IN ✓

ex. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y - x^2y^2}{\sqrt{x}-\sqrt{y}} \cdot \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \ldots$

DOMAIN :



2 PATH TEST FOR NON EXISTENCE OF A LIMIT

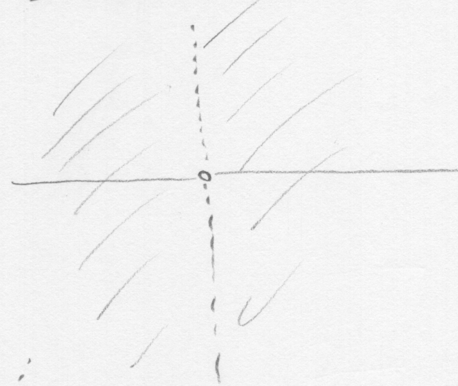
> IF A FUNCTION $f(x,y)$ HAS DIFFERENT LIMITS ALONG TWO DIFFERENT PATHS IN THE DOMAIN OF f AS $(x,y)\to(x_0,y_0)$, THEN $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ D.N.E.

ex. SHOW THAT THE LIMIT DOES NOT EXIST

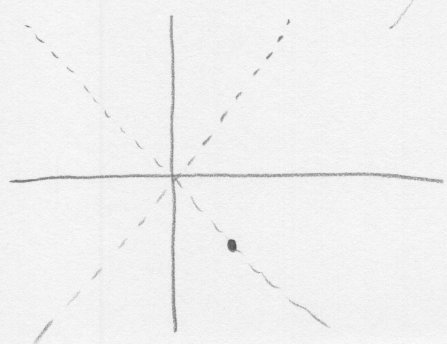
(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4-y^2}{x^4+y^2}$ ALONG $x=0$

ALONG $y=0$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{x}$ ALONG $y=mx$



(c) $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{xy+1}{x^2-y^2}$



ALONG $x=1$

$y=-1$

Def: A function $f(x,y)$ is continuous at the point (x_0,y_0) if

1. f is defined at (x_0,y_0)

2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exist.

3. $\underbrace{\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)}_{\text{exists } \checkmark} = \underbrace{f(x_0,y_0)}_{\text{is defined } \checkmark}$

Note: All of the functions we study are continuous on their domains

Except (possibly) piecewise defined functions.

e.g. Show that

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at every point except the origin.

Def: A FUNCTION $f(x,y)$ IS CONTINUOUS AT (x_0, y_0) IF

$$\underbrace{\lim_{(x,y)\to(x_0,y_0)} f(x,y)}_{\text{LIMIT EXISTS}} = \underbrace{f(x_0,y_0)}_{(x_0,y_0) \text{ IS IN DOM}(f)}$$

Note: ALL OF THE FUNCTIONS WE STUDY (WITH THE EXCEPTION OF PIECEWISE DEFINED FUNCTIONS) ARE CONTINUOUS ON THEIR DOMAINS.

e.g. LET $f(x,y) = \dfrac{2xy}{x^2+y^2}$.

DESCRIBE THE SET OF POINTS WHERE f IS CONTINUOUS.

$= \text{DOM}(f) = \{(x,y) \mid x^2+y^2 \neq 0\}$

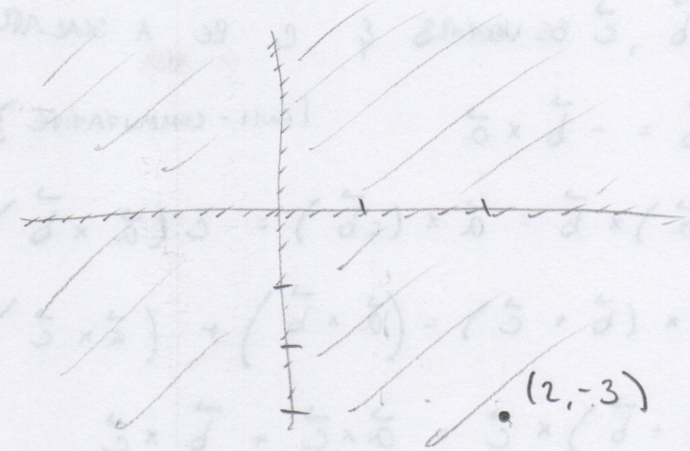
$=$ ALL POINTS EXCEPT $(0,0)$.

NOW CONSIDER $g(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{IF } (x,y) \neq (0,0) \\ 0 & \text{IF } (x,y) = (0,0) \end{cases}$

DESCRIBE THE SET OF POINTS WHERE g IS CONTINUOUS.

$$\lim_{(x,y)\to(2,-3)} \left(\frac{1}{x} + \frac{1}{y}\right)^2$$

DOMAIN



$$= \left(\frac{1}{2} + \frac{1}{-3}\right)^2$$

$$= \boxed{\frac{1}{36}}$$

<u>Def:</u> A FUNCTION $f(x,y)$ IS CONTINOUS AT (x_0, y_0)

IF

$$\underbrace{\lim_{(x,y)\to(x_0,y_0)} f(x,y)}_{\text{(1) LIMIT EXISTS.}} \overset{(3)}{=} \underbrace{f(x_0, y_0)}_{\text{(2) } (x_0, y_0) \text{ IN } \mathrm{Dom}(f)}$$

A FUNCTION $f(x,y)$ IS CONTINUOUS ON A REGION R IF IT IS CONTINUOUS AT EVERY POINT IN R.

<u>Note:</u> ALL OF THE "ELEMENTARY" FUNCTIONS WE KNOW ARE CONTINUOUS ON THEIR DOMAIN.

<u>IF YOU CAN PLUG IN</u> TO LIM, DO IT!

$$\lim_{x\to 2} \overbrace{x^2 + 3x - 1}^{\text{CONTINUOUS @ } x=2} = 9$$

PLUG IN!

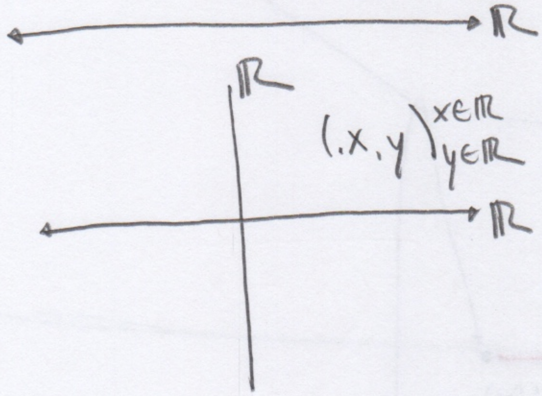
$$f(x,y) = \frac{x \sin y}{x^2+1}$$ ← DESCRIBE THE REGION WHERE f IS CONTINUOUS.

REGION OF CONTINUITY = DOMAIN $= \{(x,y) \mid x^2+1 \neq 0\}$

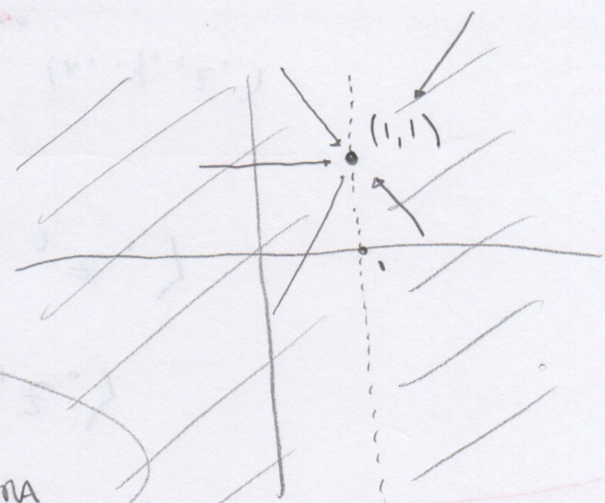
$= \{(x,y) \mid x^2 \neq -1\}$

= ENTIRE x,y PLANE

$= \mathbb{R}^2$



4 TERMS: FACTOR BY GROUPING



ex. $$\lim_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \frac{xy - y - 2x + 2}{x-1}$$

TO EVALUATE, SIMPLIFY WITH ALGEBRA, THEN PLUG IN.

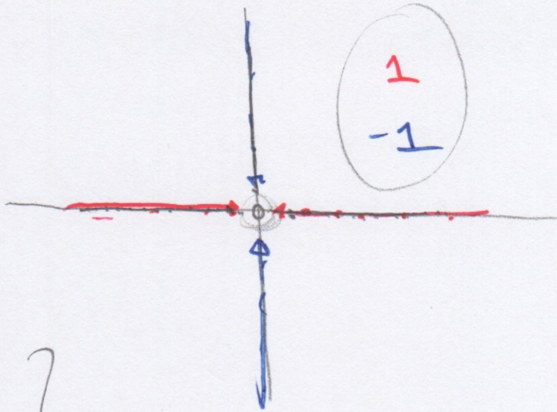
$+2(-x+1)$

$$\lim_{(x,y)\to(1,1)} \frac{y(x-1)-2(x-1)}{(x-1)} = \lim_{(x,y)\to(1,1)} \frac{\cancel{(x-1)}(y-2)}{\cancel{(x-1)}}$$

PLUGIN

$$= \lim_{(x,y)\to(1,1)} y-2 = 1-2 = \boxed{-1}$$

SHOW $\lim_{(x,y)\to(0,0)} \dfrac{x^4 - y^2}{x^4 + y^2}$ DNE



ALONG X-AXIS $(y=0)$ $\lim_{x\to 0} \dfrac{x^4}{x^4} = 1$

ALONG Y-AXIS $(x=0)$ $\lim_{y\to 0} \dfrac{-y^2}{y^2} = -1$

2 DIFF LIMITS ALONG 2 DIFF. PATHS TO $(0,0)$.

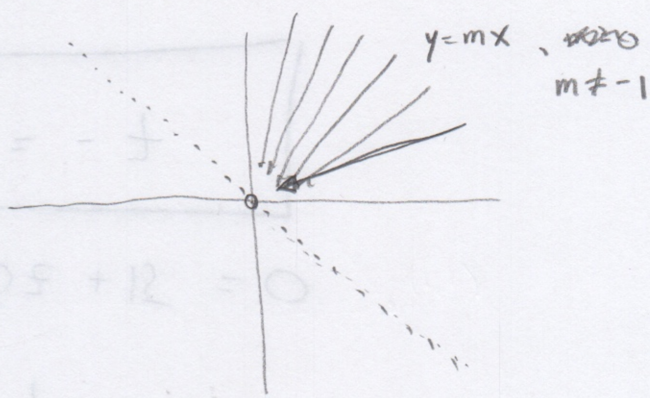
$\Rightarrow$ LIMIT D.N.E.

$\lim_{(x,y)\to(0,0)} \dfrac{x-y}{x+y}$

SHOW LIM D.N.E.

$x+y \neq 0$

$y \neq -x$



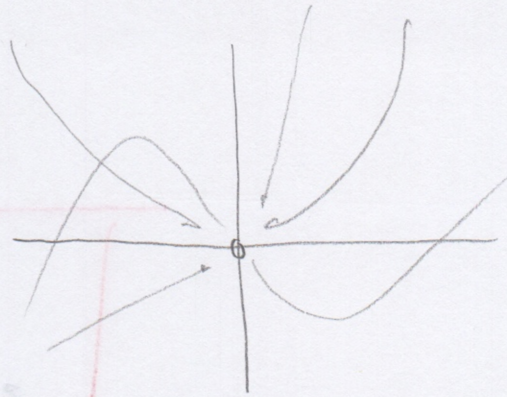
$y = mx$ $\quad \lim_{x\to 0} \dfrac{x - mx}{x + mx} = \lim_{x\to 0} \dfrac{\cancel{x}(1-m)}{\cancel{x}(1+m)}$

$(y = mx \to 0)$

$= \dfrac{1-m}{1+m}$ DIFFERENT LIMITS ALONG LINES WITH DIFF. SLOPES.

$m = \frac{1}{2}$ $\quad \dfrac{1-\frac{1}{2}}{1+\frac{1}{2}} = \boxed{\frac{1}{3}}$

ex. $$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$$



TRY THESE PATHS: X-AXIS $(y=0)$ $\to 0$

$\boxed{\text{Y-AXIS } (x=0)}$ $\to 0$

$y = x$

$\boxed{y = mx}$

ALSO: $\underbrace{y = x^2, \quad y = x^n}_{\lim_{x\to 0}}$

$\underbrace{x = y^2, \quad x = y^n}_{\lim_{y\to 0}}$

$$y = mx, \quad \lim_{x\to 0} \frac{x^2(mx)}{x^4 + (mx)^2}$$

$$= \lim_{x\to 0} \frac{mx^3}{x^4 + m^2x^2} = \lim_{x\to 0} \frac{\cancel{x^2}(mx)}{\cancel{x^2}(x^2 + m^2)} = \frac{0}{m^2} = 0$$

$$y = x^2 \quad \lim_{x\to 0} \frac{x^2(x^2)}{x^4 + (x^2)^2} = \lim_{x\to 0} \frac{x^4}{x^4 + x^4} = \lim_{x\to 0} \frac{\cancel{x^4}}{2\cancel{x^4}} = \frac{1}{2}$$

DIFF LIMITS ALONG DIFF. PATHS TO $(0,0)$ $\Longrightarrow$ $\lim_{(x,y)\to(0,0)}$ D.N.E

To show $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ EXIST

$\underbrace{\qquad\qquad\qquad}$

USE ALGEBRA TO SIMPLIFY → PLUG IN.

FACTORING

MULT. TOP & BOT BY RADICAL CONJ.

To show $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ D.N.E.

FIND 2 PATHS TO (x_0, y_0) S.T. LIMITS ALONG THE PATHS ARE DIFFERENT.

$y = c = 0$

$x = c = 0$

$y = mx$

$x = y^2$

$$\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2}$$

ALONG X-AXIS $(y=0) \longrightarrow 0$

ALONG Y-AXIS $(x=0) \longrightarrow 1$

$\Big\}$ LIMIT D.N.E.

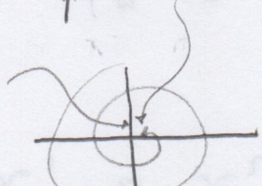
$$\lim_{(x,y)\to(0,0)} \frac{x^3 - xy^2}{x^2+y^2}$$

Y-AXIS $(x=0) \longrightarrow 0$

X-AXIS $(y=0) \longrightarrow \lim\limits_{x\to 0} \dfrac{x^2(x)}{x^2} = 0$

$y = x \longrightarrow 0$

POLAR COORD!



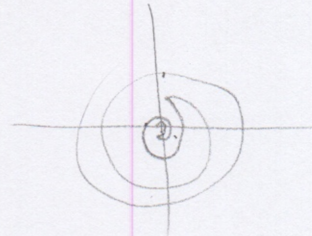
$(x,y) \to (0,0)$

$r \longrightarrow 0$

REPLACE x WITH $r\cos\theta$

y WITH $r\sin\theta$

AND LET $r \to 0$.



$$\lim_{\substack{r\to 0\\(\theta=?)}} \frac{(r\cos\theta)^3-(r\cos\theta)(r\sin\theta)^2}{(r\cos\theta)^2+(r\sin\theta)^2}$$

$$= \lim_{r\to0} \frac{\cancel{r^3}\cos\theta\overbrace{(\cos^2\theta-\sin^2\theta)}^{\cos 2\theta}}{\cancel{r^2}\underbrace{(\cos^2\theta+\sin^2\theta)}_{1}}$$

$$=\lim_{\substack{r\to0\\\theta=?}} \frac{r\overbrace{\cos\theta\cos2\theta}^{-1\le * \le 1}}{1} =$$

$$\underbrace{\lim_{r\to0} -r}_{0} \le \lim_{r\to0} r\cos\theta\cos2\theta \le \underbrace{\lim_{r\to0} r}_{0}$$

$$0 \le \qquad \downarrow \qquad \le 0$$

$$0.$$

IF $\lim_{r\to0} f(r\cos\theta, r\sin\theta)$

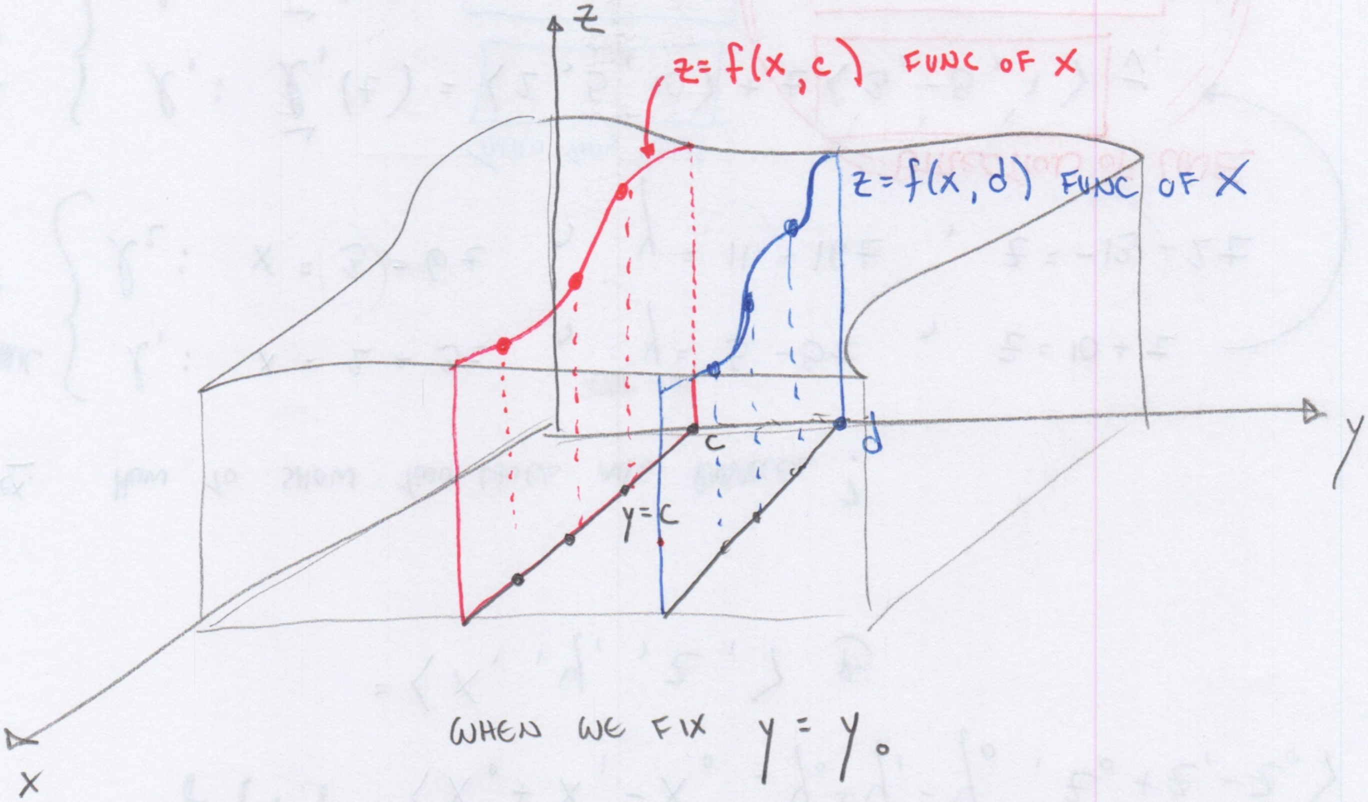
EXISTS THEN $\lim_{(x,y)\to(0,0)} f(x,y)$

EXIST

(& SAME LIMIT)

§ 14.3 PARTIAL DERIVATIVES

$z = f(x,y)$ GRAPH/ SURFACE



WHEN WE FIX $y = y_0$

$z = f(x, y_0)$ IS NOW A FUNC OF 1 VAR, X.

NOT A VARIABLE (CONSTANT)

WE GET A DIFF. FUNC. OF X

FOR EVERY VALUE OF y.

EVERY TIME YOU CHANGE THE VALUE y,

YOU GET A NEW FUNC. OF X.