

§ 10.5 EQUATIONS OF LINES & PLANES

LINES

LINE THROUGH $\vec{r}_0$ IN DIRECTION $\vec{v}$:

VECTOR EQUATION: $\vec{r} = \vec{r}_0 + t\vec{v}, \quad -\infty < t < \infty$

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$$

$$\Rightarrow x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

PARAMETRIC EQ'S:

↓ SOLVE FOR t

SYMMETRIC
EQ'S:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

(ASSUMING $a, b, c \neq 0$)

↑

IF ONE IS ZERO, WE STILL

SOLVE FOR t IN REMAINING PARAM.
EQ'S.

e.g. IF $a = 0$

$$x = x_0; \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

EXAMPLE

THE LINE SEGMENT FROM $\vec{r}_0$ TO $\vec{r}_1$ IS GIVEN BY

$$\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1, \quad 0 \leq t \leq 1$$

NOTE: $\vec{r}(0) = \vec{r}_0, \quad \vec{r}(1) = \vec{r}_1$

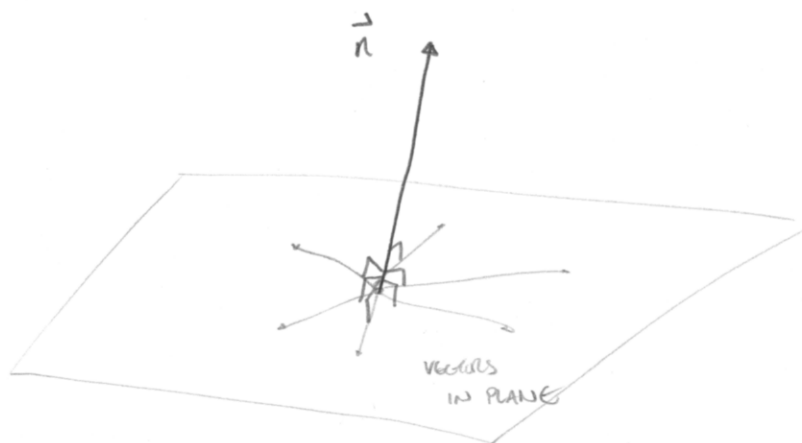
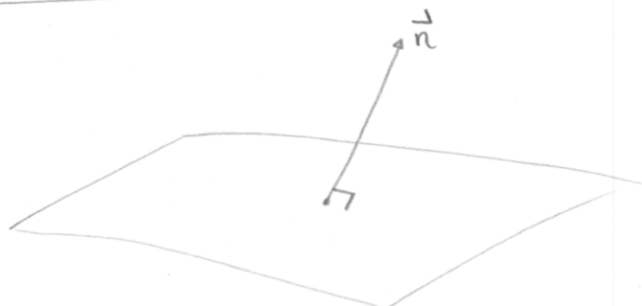
e.g. SHOW THAT LINES L_1 & L_2 ARE SKW LINES (not $\parallel$, don't intersect)

$$L_1: x = 1+t, y = -2+3t, z = 4-t$$

$$L_2: x = 2s, y = 3+s, z = -3+4s$$

PLANES

Normal vector $\vec{n}$



IF $\vec{r}_0$ IS IN THE PLANE & $\vec{n}$ IS NORMAL TO PLANE,

THEN $\vec{r}$ IS IN THE PLANE $\Leftrightarrow \underbrace{\vec{r} - \vec{r}_0}_{\text{vector in the plane}} \perp \vec{n}$

VECTOR EQ:

$$\text{i.e. } \vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \quad \text{or} \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

$$\vec{n} = \langle a, b, c \rangle$$

SCALAR EQ:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz + d = 0$$

EXAMPLES

Point,
Normal vector

PLANE THROUGH
3 PTS

NOTE: THE ANGLE BETWEEN TWO PLANES IS DEFINED TO BE THE ACUTE ANGLE BETWEEN THEIR NORMAL VECTORS.

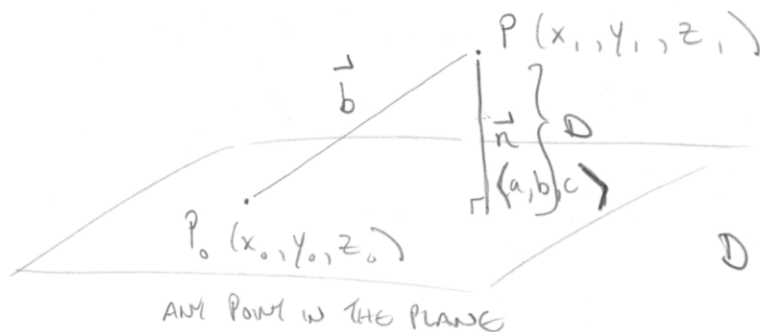
e.g. $\parallel$ PLANES HAVE $\parallel$ NORMAL VECTORS.

e.g. FIND ANGLE BETWEEN PLANES

$$\left(\text{RECALL } \vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1| |\vec{n}_2| \cos \theta \right)$$

EXAMPLE : p. 576 #6.b.

e.g. FIND DISTANCE FROM $P(x_1, y_1, z_1)$ TO PLANE $ax + by + cz + d = 0$.



$$D = \left| \text{comp}_{\vec{n}} \vec{b} \right| = \frac{|\vec{n} \cdot \vec{b}|}{|\vec{n}|}$$

$$D = \frac{|ax_1 + by_1 + cz_1|}{\sqrt{a^2 + b^2 + c^2}}$$

e.g. FIND DISTANCE BETWEEN 2 $\parallel$ PLANES.

§ 12.5 EQUATIONS OF LINES & PLANES

LINES:

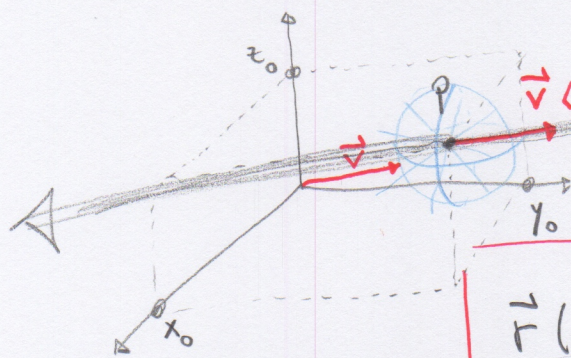
IN

3D

~~SLOPE~~

HAVE

DIRECTION



Point on the line $P(x_0, y_0, z_0)$

Direction $\parallel$ to line $\vec{v}\langle a, b, c \rangle$

Vector
Eq :

$$\vec{r}(t) = \vec{P} + t\vec{v}, \quad -\infty < t < \infty$$

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

3 Parametric EQ's :

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct, \quad -\infty < t < \infty$$

Solve for t

$$t = \frac{x - x_0}{a}, \quad t = \frac{y - y_0}{b}, \quad t = \frac{z - z_0}{c}$$

$\Rightarrow$

Symmetric
EQ's :

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

THROUGH THE POINTS $(-3, 5, 4)$ & $(4, -7, 2)$

A B

$\vec{r}(t) = \vec{r}_0 + t\vec{v}$

↑
 VECTOR TERMINATES ON
 THE LINE

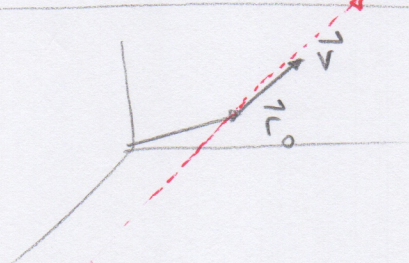
PARALLEL TO LINE

$\vec{AB} = \langle 4 - (-3), -7 - 5, 2 - 4 \rangle = \vec{v}$

CHOOSE $\langle -3, 5, 4 \rangle = \vec{r}_0$

$$\vec{r}(t) = \langle -3, 5, 4 \rangle + t \langle 7, -12, -2 \rangle, -\infty < t < \infty$$

VECTOR EQ.



PARAMETRIC EQ'S

$$x = -3 + 7t, \quad y = 5 - 12t, \quad z = 4 - 2t$$

$$-\infty < t < \infty$$

SYMM.
EQ'S

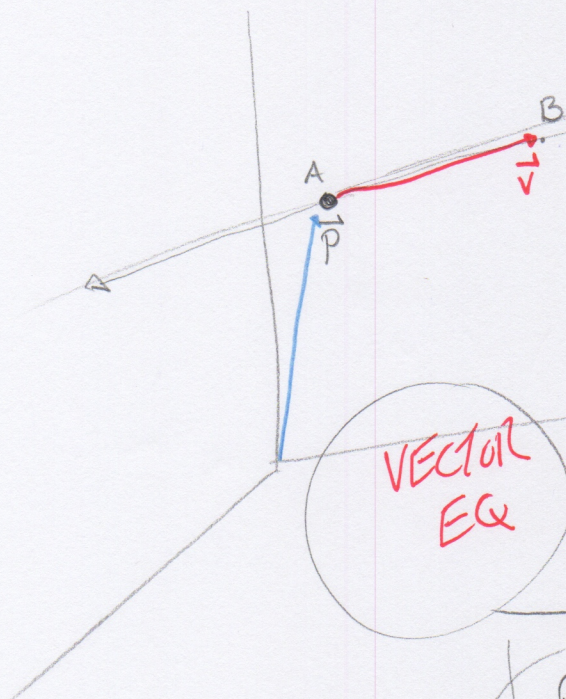
$$\frac{x+3}{7} = \frac{y-5}{-12} = \frac{z-4}{-2}$$

SYMMETRIC EQ'S

ex. GIVE EQ FOR THE LINE THAT CONTAINS $A(5, 1, -3)$, $B(0, -4, 2)$

POINT ON LINE : $\vec{p} = \langle 5, 1, -3 \rangle$

DIRECTION FOR LINE : $\vec{v} = \overrightarrow{AB} = \langle 0-5, -4-1, 2-(-3) \rangle$
 $= \langle -5, -5, 5 \rangle$



$$\vec{r}(t) = \vec{p} + t\vec{v}$$

$$\vec{r}(t) = \langle 5, 1, -3 \rangle + t\langle -5, -5, 5 \rangle$$

VECTOR
EQ

$$\vec{r}(t) = \langle 5-5t, 1-5t, -3+5t \rangle$$

PARAMETRIC;
EQ

$$x = 5 - 5t$$

$$y = 1 - 5t$$

$$z = -3 + 5t$$

ex. LINE SEGMENT FROM $A(5, 1, -3)$ TO $B(0, -4, 2)$

$$\vec{r}(t) = \vec{A} + t\overrightarrow{AB}, \quad 0 \leq t \leq 1$$

ex.

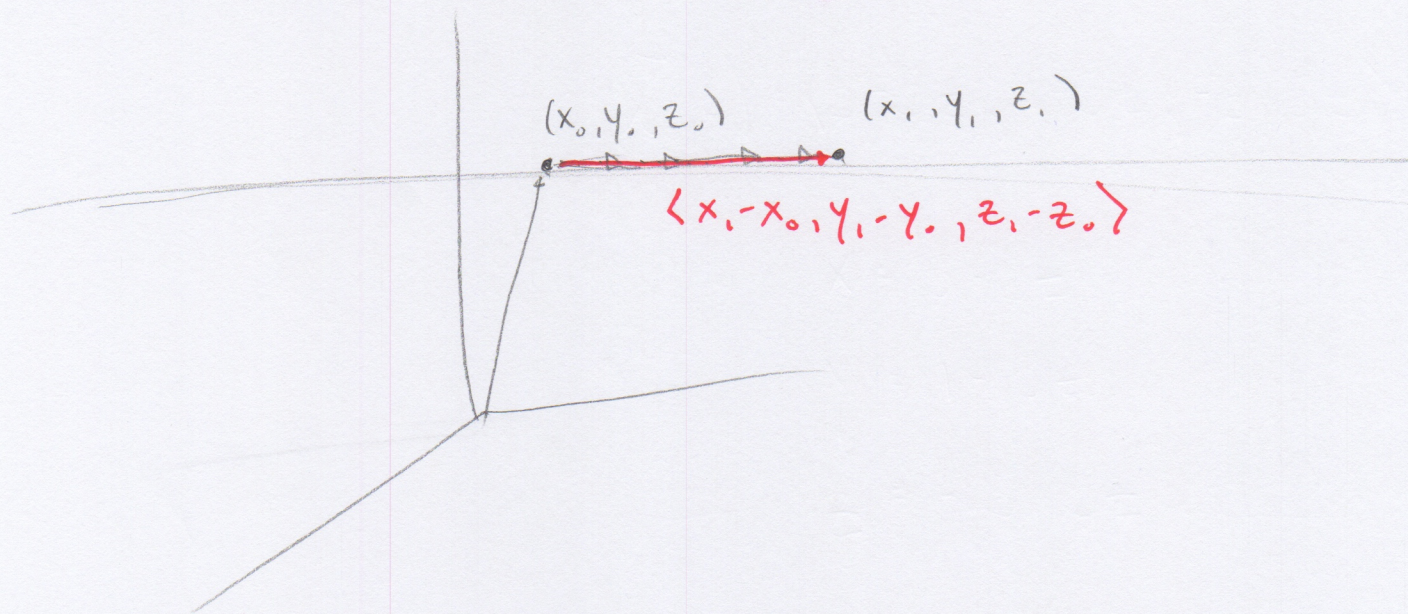
LINE SEGMENT FROM (x_0, y_0, z_0) TO (x_1, y_1, z_1)

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

$$0 \leq t \leq 1$$

VERIFY: $\vec{r}(0) = \langle x_0, y_0, z_0 \rangle$

$$\vec{r}(1) = \langle x_1, y_1, z_1 \rangle$$



LINE SEGMENT FROM $A(x_0, y_0, z_0)$ TO $B(x_1, y_1, z_1)$

Let $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

Let $\vec{v} = \vec{AB} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

$\vec{r}(t) = \vec{r}_0 + t\vec{v}, \quad 0 \leq t \leq 1$

$\vec{r}(0) = \langle x_0, y_0, z_0 \rangle \quad A$

$\vec{r}(1) = \langle x_0 + x_1 - x_0, y_0 + y_1 - y_0, z_0 + z_1 - z_0 \rangle$
 $= \langle x_1, y_1, z_1 \rangle \quad B$

ex. How to show two lines are parallel?

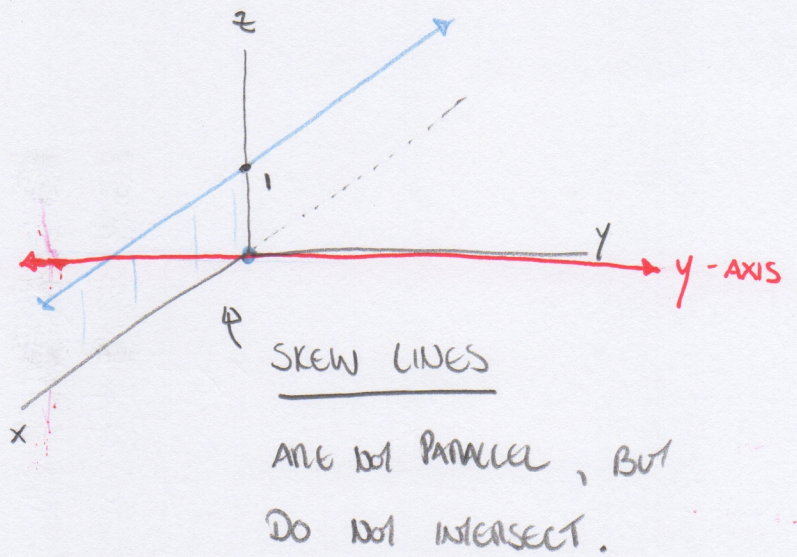
PARAMETRIC EQ'S $\begin{cases} l_1: x = 2 + 3t, & y = 5 - 8t, & z = 10 + t \\ l_2: x = 3 - 6t, & y = 11 + 16t, & z = -13 - 2t \end{cases}$

VECTOR EQ'S $\begin{cases} l_1: \vec{r}_1(t) = \langle 2, 5, 10 \rangle + t \langle 3, -8, 1 \rangle \vec{v}_1 \\ l_2: \vec{r}_2(t) = \langle 3, 11, -13 \rangle + t \langle -6, 16, -2 \rangle \vec{v}_2 \end{cases}$

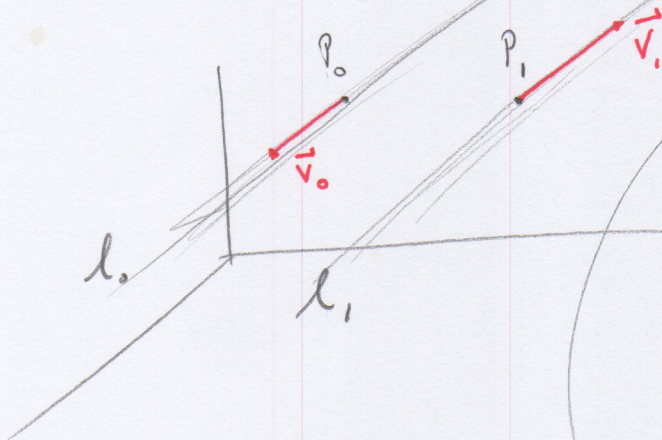
$-2\vec{v}_1 = \vec{v}_2$
 $-\frac{1}{2}\vec{v}_2 = \vec{v}_1$

EACH VECTOR IS A SCALAR MULTIPLE OF THE OTHER.

WHEN ARE 2 LINES PARALLEL?



2 LINES ARE PARALLEL WHEN THEY HAVE DIRECTION VECTORS THAT POINT IN THE SAME (OR OPPOSITE) DIRECTIONS.



$l_0 \parallel l_1$ WHEN

VECTORS $\vec{v}_0$ & $\vec{v}_1$

ARE SCALAR MULTIPLES OF EACH OTHER.

e.g.

e.x.

$$l_1: x = 8 + 2t, y = 10 - 3t, z = -7 + 5t$$

$$l_2: x = -2 - 4t, y = -6 + 6t, z = 11 - 10t$$

Are l_1 & $l_2 \parallel$?

$$\vec{l}_1(t) = \langle 8, 10, -7 \rangle + t \langle 2, -3, 5 \rangle$$

$$\vec{l}_2(t) = \langle -2, -6, 11 \rangle + t \langle -4, 6, -10 \rangle$$

$$-2t \langle 2, -3, 5 \rangle$$

DIRECTION VECTORS
ARE MULTIPLES
(SCALAR) OF
EACH OTHER.

ex.

GIVE PARAMETRIC EQ'S FOR THE LINE THROUGH $(1, -2, 5)$

PARALLEL TO THE LINE GIVEN PARAMETRICALLY BY

$$x = 8 + 5t$$

$$y = 1 - 4t$$

$$z = -3 + 3t$$

$$x = 1 + 5t$$

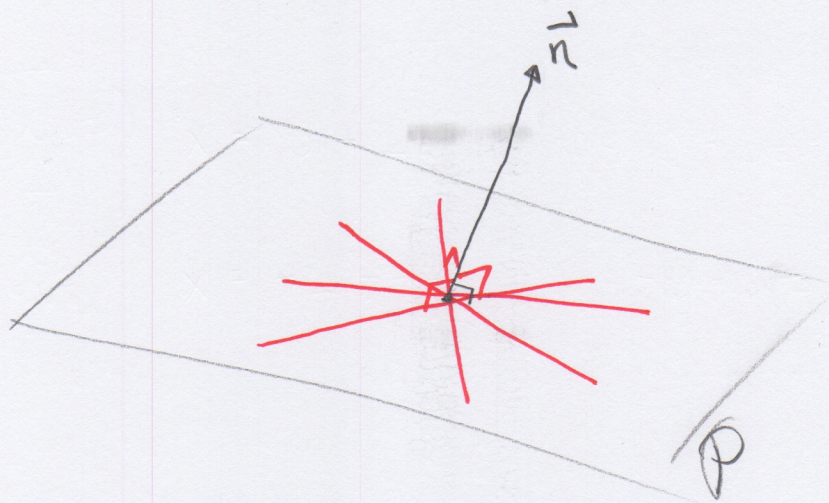
$$y = -2 - 4t$$

$$z = 5 + 3t$$

$$\vec{l}(t) = \langle 8, 1, -3 \rangle + t \underline{\underline{\langle 5, -4, 3 \rangle}}$$

$$\vec{r}(t) = \langle 1, -2, 5 \rangle + t \underline{\underline{\langle 5, -4, 3 \rangle}}$$

PLANES

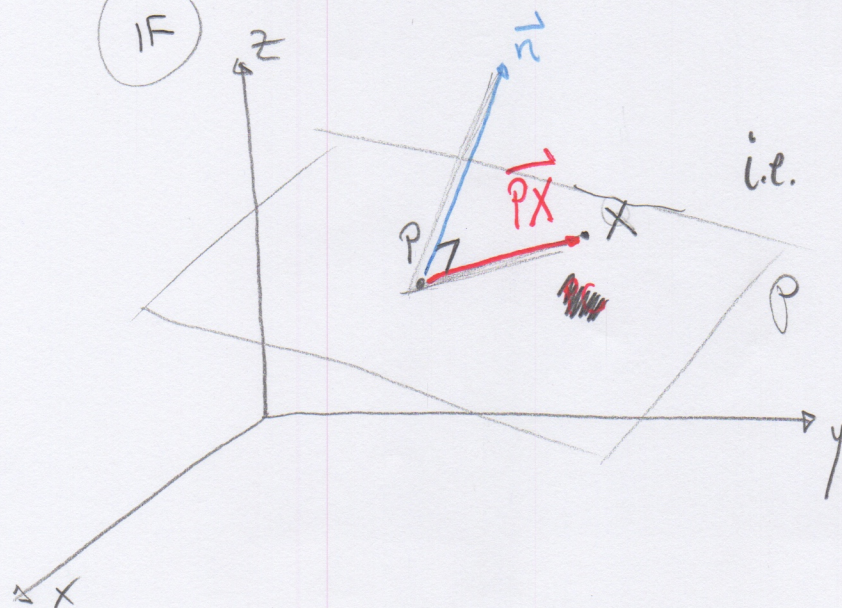


GIVEN A PLANE P , A VECTOR $\vec{n}$ IS CALLED
A NORMAL VECTOR TO THE PLANE IF IT IS
PERPENDICULAR TO EVERY VECTOR IN THE PLANE.
(i.e. PERPENDICULAR TO THE PLANE)

IF $\vec{F}$ IS A VECTOR THAT
Let $P(x_0, y_0, z_0)$ BE A POINT IN THE PLANE P .
AND Let $\vec{n}$ BE A NORMAL VECTOR TO THE PLANE P .

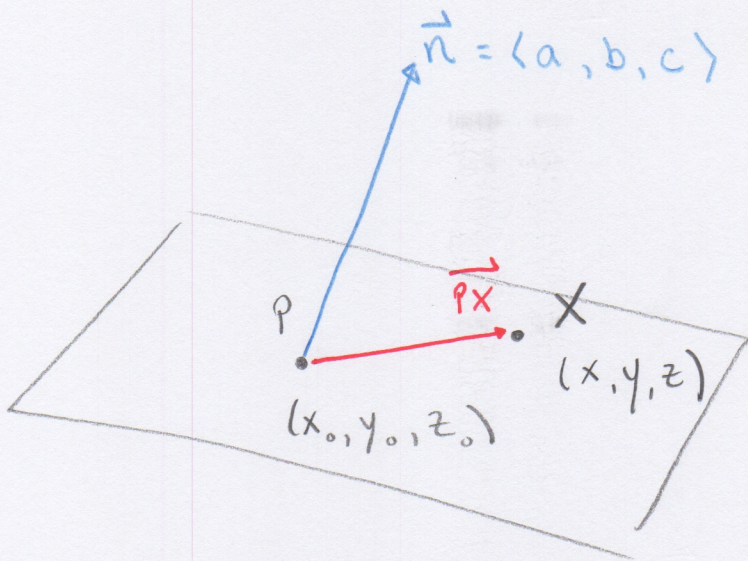
THEN $X(x, y, z)$ IS IN THE PLANE P

(IF



$$\vec{PX} \perp \vec{n}.$$

$$\text{i.e. } \vec{PX} \cdot \vec{n} = 0$$



PLANE THROUGH (x_0, y_0, z_0) WITH
NORMAL VECTOR $\vec{n} = \langle a, b, c \rangle$.

HAS EQ:

$$\overrightarrow{PX} \cdot \vec{n} = 0$$

$$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$$

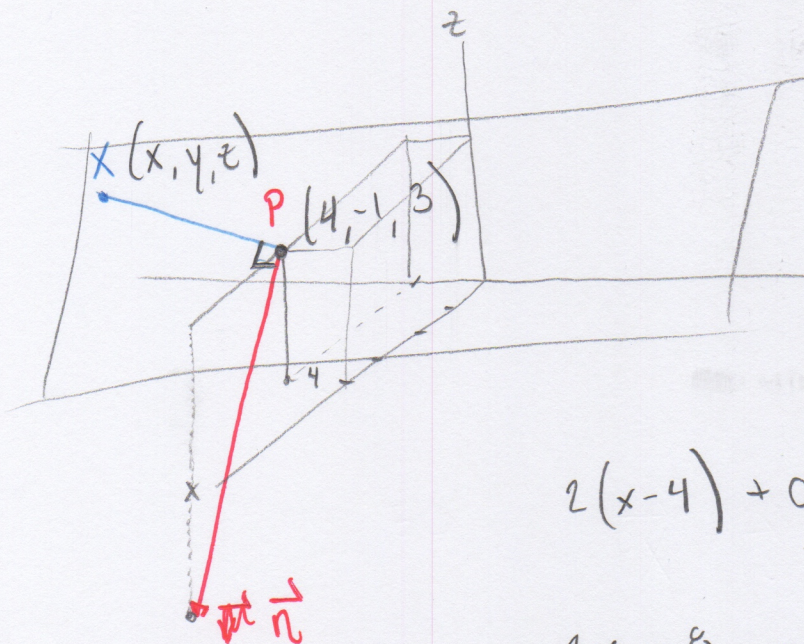
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}_{\text{CONSTANT } d}$$

$$ax + by + cz = d$$

FIND EQ. OF PLANE THROUGH $(4, -1, 3)^P$

PERPENDICULAR TO VECTOR $\langle 2, 0, -5 \rangle \vec{n}$



$$\vec{PX} \cdot \vec{n} = 0$$

$$\langle x-4, y+1, z-3 \rangle \cdot \langle 2, 0, -5 \rangle = 0$$

$$2(x-4) + 0(y+1) - 5(z-3) = 0$$

$$2x - 8 - 5z + 15 = 0$$

$$2x - 5z = -7$$