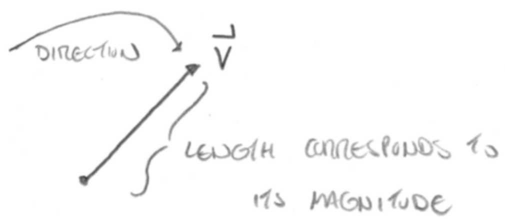
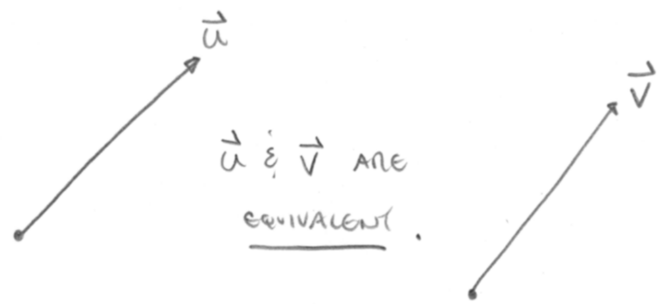
§10.2 VECTORS

A VECTOR IS A QUANTITY WITH BOTH MAGNITUDE (SIZE) & DIRECTION.

e.g. DISPLACEMENT, VELOCITY, FORCE, ETC...

DENOTE BY BOLDFACE TYPE OR $\rightarrow$ ABOVE THE LETTER (e.g. VECTOR **v** or $\vec{v}$).

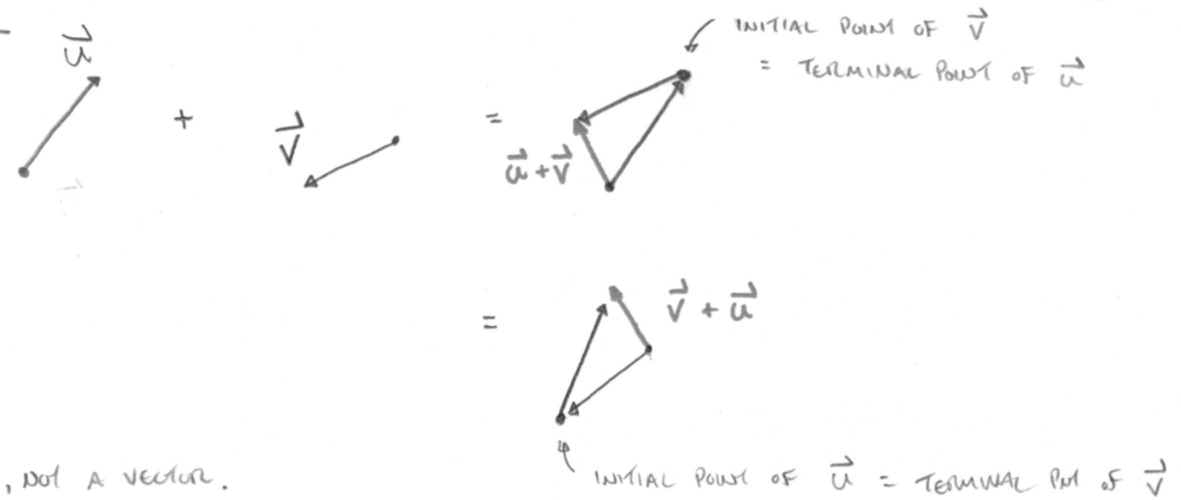




<u>NOTE:</u> 0 VECTOR DOES NOT HAVE SPECIFIC DIRECTION.

THE GEOMETRY

ADDING VECTORS:

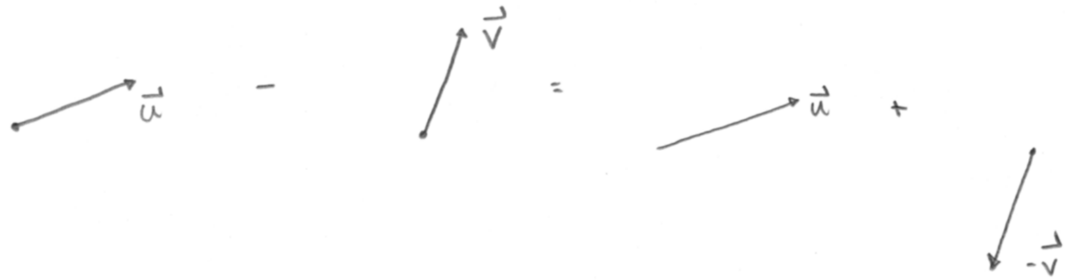


REAL #, NOT A VECTOR.

MULTIPLYING BY SCALAR: LET c BE A SCALAR & $\vec{v}$ BE A VECTOR, THEN $c\vec{v}$ IS THE VECTOR WHOSE MAGNITUDE (LENGTH) IS $|c|$ TIMES THE MAGNITUDE (LENGTH) OF $\vec{v}$ & WHOSE DIRECTION IS THE SAME AS $\vec{v}$ IS $c > 0$, OPPOSITE $\vec{v}$ IF $c < 0$, AND $c\vec{v} = 0$ IF EITHER $\vec{v} = 0$ OR $c = 0$.

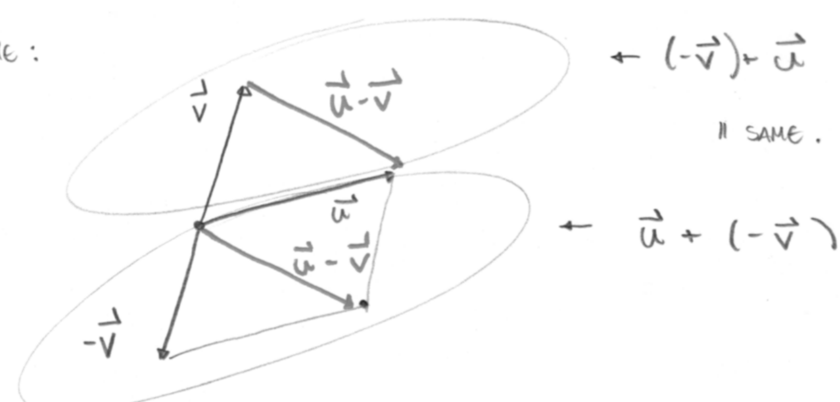
e.g.

SUBTRACTION : $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$



$=$

NOTE :

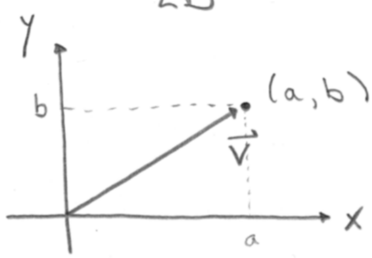


THE ALGEBRA

GIVEN A VECTOR, WE CAN INTRODUCE A COORDINATE SYSTEM IN ORDER TO DESCRIBE IT PRECISELY BY CALLING ITS INITIAL POINT THE ORIGIN & CALLING ITS TERMINAL POINT BY ITS COORDINATES, CALLED THE <u>COMPONENTS</u> OF THE VECTOR.

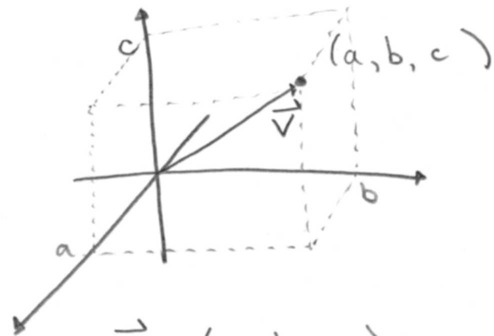
e.g.

2D



$\vec{v} = \langle a, b \rangle$

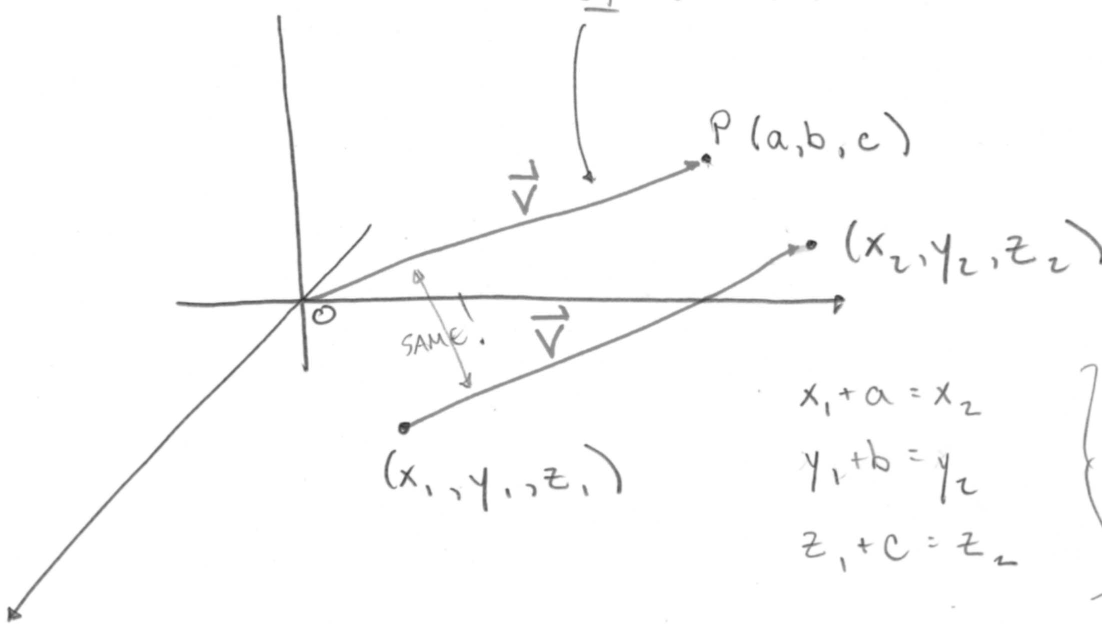
3D



$\vec{v} = \langle a, b, c \rangle$

NOTATION!

Def: Position Vector (initial pnt origin)



$$\left.\begin{array}{l} x_1 + a = x_2 \\ y_1 + b = y_2 \\ z_1 + c = z_2 \end{array}\right\} \rightarrow \begin{array}{l} x_2 - x_1 = a \\ y_2 - y_1 = b \\ z_2 - z_1 = c \end{array}$$

GIVEN $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$, THE VECTOR $\vec{v}$ WITH REPRESENTATION $\overrightarrow{AB}$ IS GIVEN BY

$$\vec{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

e.g. FIND VECTOR REPRESENTED BY THE DIRECTED LINE SEGMENT FROM $A(3, -7, 5)$ TO $B(5, 3, -6)$.

THE **MAGNITUDE** (**LENGTH**) OF A VECTOR $\vec{v} = \langle x, y \rangle$ OR $\vec{v} = \langle x, y, z \rangle$ IS DENOTED $|\vec{v}|$ OR $\|\vec{v}\|$ AND IS GIVEN BY

$$|\vec{v}| = \sqrt{x^2 + y^2} \quad \text{OR} \quad |\vec{v}| = \sqrt{x^2 + y^2 + z^2}$$

NOTE THAT THIS FOLLOWS FROM THE DISTANCE FORMULA.

NOTE THAT
$|\vec{v}| \geq 0$ & $|\vec{v}| = 0 \iff \vec{v} = \vec{0}$

IF $\vec{a} = \langle a_1, a_2 \rangle$ & $\vec{b} = \langle b_1, b_2 \rangle$

THEN
$$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$$
$$\vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$
$$c\vec{a} = \langle ca_1, ca_2 \rangle$$

& FOR 3D AS WELL.

e.g. $\langle 3, -4 \rangle - 5\langle -6, 8 \rangle = \langle 3 - 5(-6), -4 - 5(8) \rangle = \langle 33, -44 \rangle$

PROPERTIES OF VECTORS

IF $\vec{a}, \vec{b}, \vec{c}$ ARE n-DIMENSIONAL VECTORS & c, d ARE SCALARS THEN

(i) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (COMM.)

(ii) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (ASS.)

(iii) $\vec{a} + \vec{0} = \vec{a}$ (ADD ID.)

(iv) $\vec{a} - \vec{a} = \vec{0}$ (ADD INVERSE.)

(v) $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$ (DIST.)

(vi) $(\vec{c} + d)\vec{a} = c\vec{a} + d\vec{a}$ (DIST.)

(vii) $(cd)\vec{a} + c(d\vec{a})$ (ASSOC.)

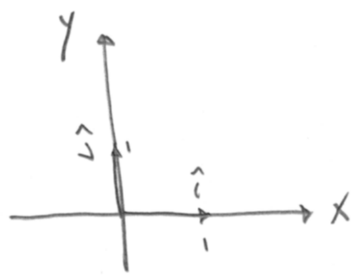
(viii) $1\vec{a} = \vec{a}$ (MULT. ID.)

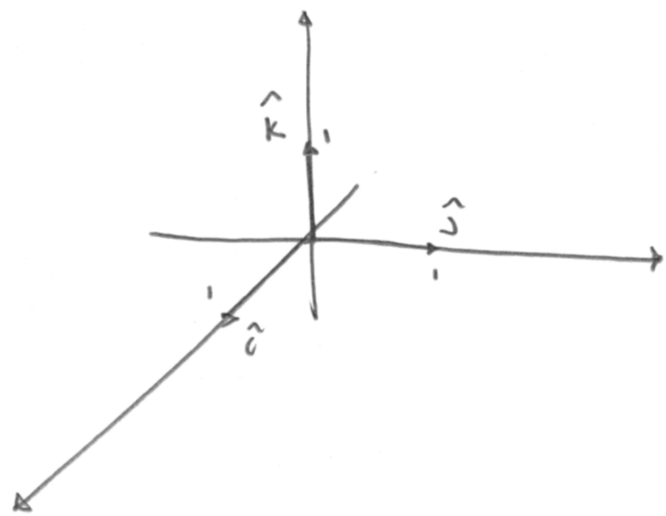
STANDARD BASIS VECTORS:

$$\hat{i} = \langle 1, 0, 0 \rangle \quad , \quad \hat{j} = \langle 0, 1, 0 \rangle \quad , \quad \hat{k} = \langle 0, 0, 1 \rangle$$

POINT IN DIR OF X-AXIS, Y-AXIS, Z-AXIS.

ALL LENGTH 1 (UNIT VECTORS).





THUS $\vec{v} = \langle x, y, z \rangle = \langle x, 0, 0 \rangle + \langle 0, y, 0 \rangle + \langle 0, 0, z \rangle$

$$= x\langle 1, 0, 0 \rangle + y\langle 0, 1, 0 \rangle + z\langle 0, 0, 1 \rangle$$

$$= x\hat{i} + y\hat{j} + z\hat{k}$$

(IN 2D) $\vec{v} = \langle x, y \rangle = x\hat{i} + y\hat{j}$

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e.g. IF $\vec{a} = 7\hat{i} + 2\hat{j} + 0\hat{k}$ & $\vec{a} = 2\hat{i} - 4\hat{j} - 11\hat{k}$

THEN $3\vec{a} + 4\vec{b} = 29\hat{i} - 10\hat{j} - 20\hat{k}$.

THE <u>UNIT VECTOR</u> THAT POINTS IN THE SAME DIRECTION AS $\vec{v}$ IS $\dfrac{1}{|\vec{v}|}\vec{v}$.

TO SEE THIS, FIRST NOTE THAT FOR SCALAR c, WE HAVE

$$
\begin{aligned}
|c\vec{v}| &= |\langle cx, cy, cz \rangle| \\
&= \sqrt{(cx)^2 + (cy)^2 + (cz)^2} \\
&= \sqrt{c^2(x^2+y^2+z^2)} = |c|\sqrt{x^2+y^2+z^2} \\
&= |c||\vec{v}|
\end{aligned}
$$

$$\therefore \left|\frac{1}{|\vec{v}|}\vec{v}\right| = \left|\frac{1}{|\vec{v}|}\right| |\vec{v}| = \frac{1}{|\vec{v}|}|\vec{v}| = 1. \checkmark$$

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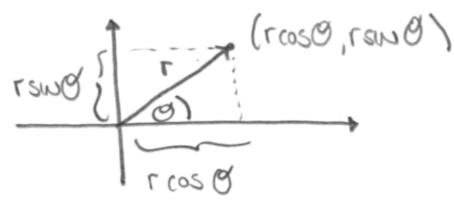
e.g. FIND UNIT VECTOR IN SAME DIRECTION AS $\langle -1, 12, -12 \rangle$.

$\left(\langle -\frac{1}{17}, \frac{12}{17}, -\frac{12}{17} \rangle \right)$

RECALL FROM POLAR COORDINATES:



NOW IF $\vec{v} = \langle x, y \rangle$ THEN

$x = |\vec{v}| \cos\theta$

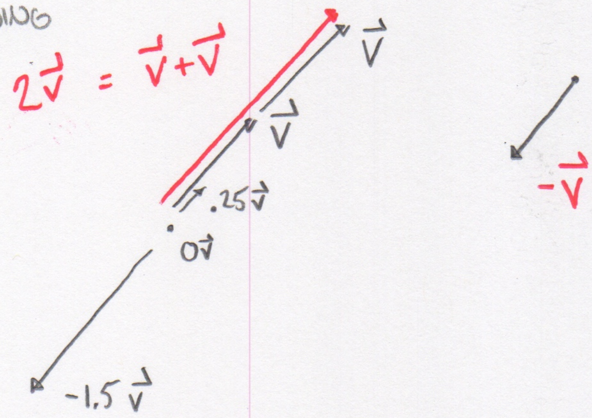
$y = |\vec{v}| \sin\theta$

WHERE θ IS ANGLE BETWEEN $\hat{i}$ & $\vec{v}$.

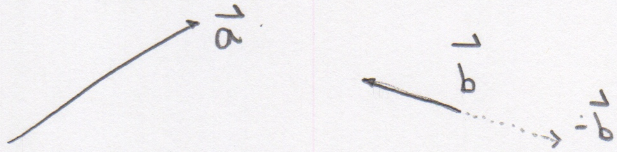
[EXAMPLE LIKE #28, OR EX 7 P. 554]

§12.2 CONTINUED VECTORS

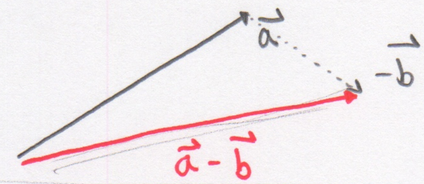
ADDING



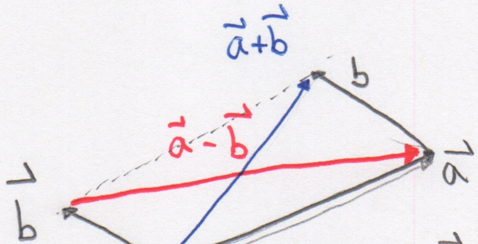
SUBTRACTING VECTORS:



$$\vec{a} - \vec{b} = \vec{a} + (-1)\vec{b}$$



$\vec{a} - \vec{b}$



BEGIN WITH $\vec{a}$

THEN MOVE INITIAL POINT OF

THAT VECTOR $(\vec{a})$ THRU $\vec{b}$

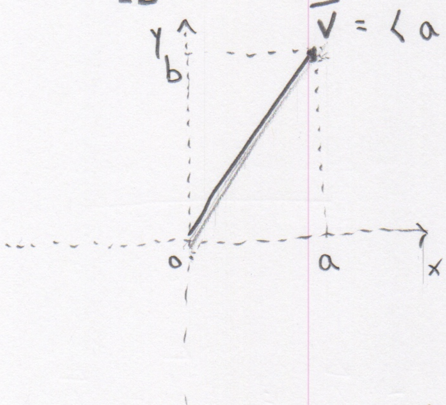
ANALYSIS OF VECTORS:

GIVEN A VECTOR, WE CAN INTRODUCE A COORDINATE SYSTEM IN ORDER TO DESCRIBE IT PRECISELY BY MAKING ITS INITIAL POINT THE ORIGIN, AND CALLING ITS TERMINAL POINT BY ITS COORDINATES, CALLED THE <u>COMPONENTS</u> OF THE VECTOR.

VECTOR

2D

$\vec{v} = \langle a, b \rangle$



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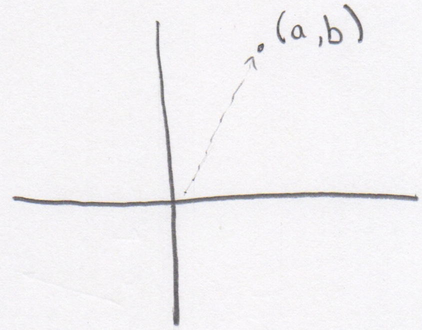
ANGLED BRACKETS FOR VECTORS.

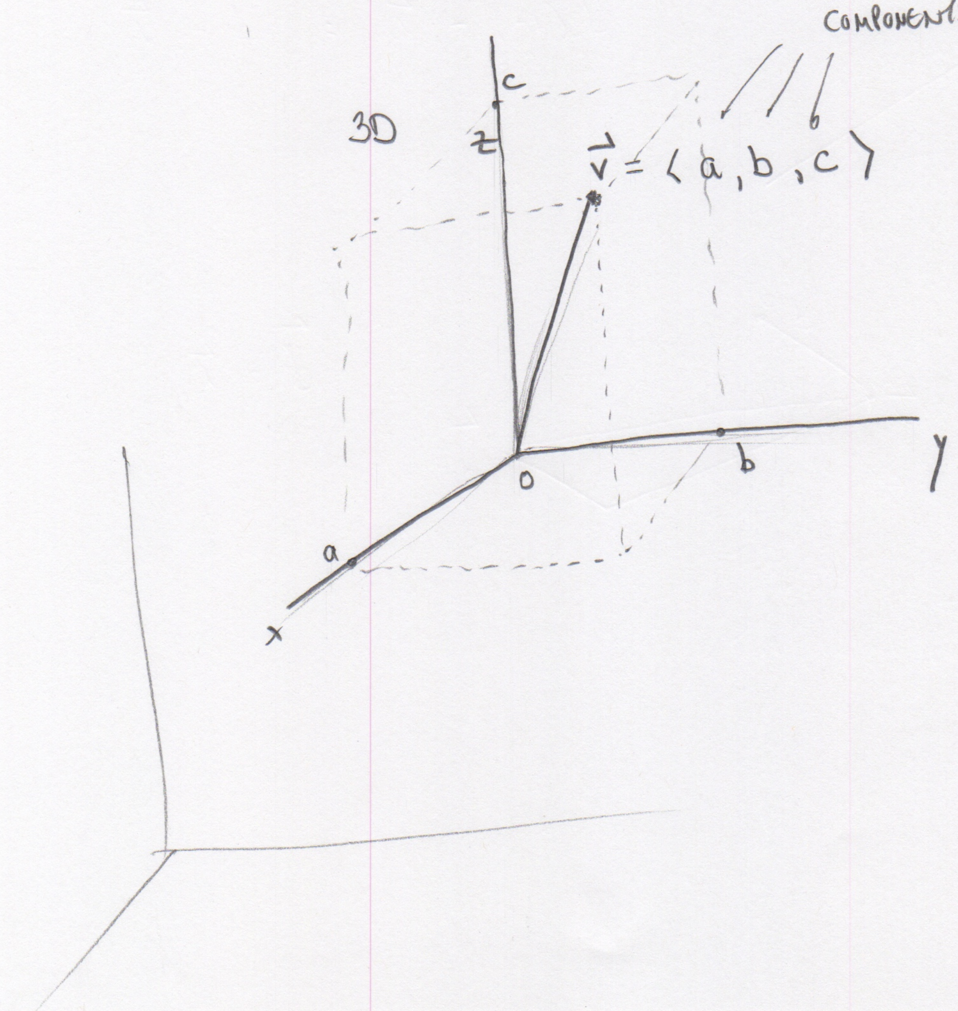
POINT



COMPONENTS OF $\vec{v}$

3D

$\vec{v} = \langle a, b, c \rangle$

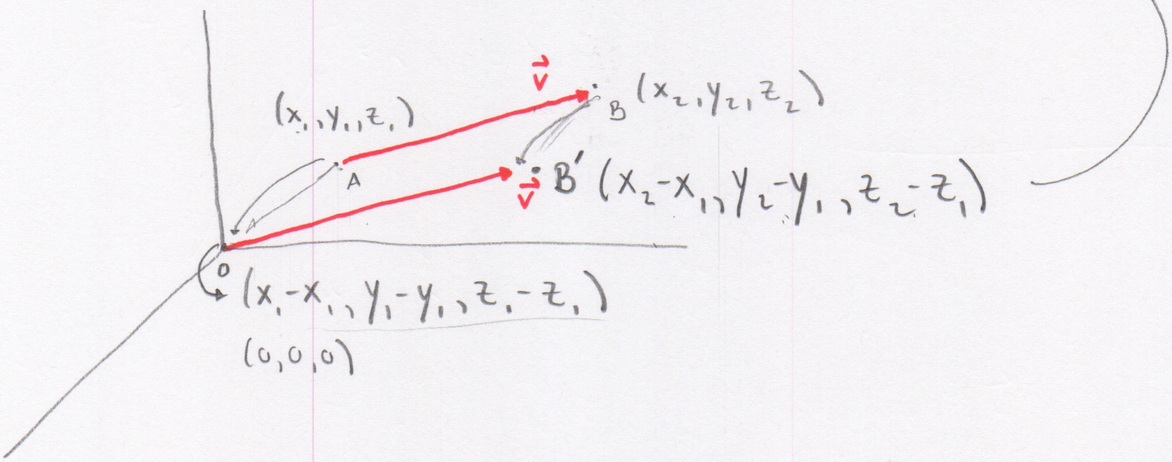


GIVEN TWO POINTS, $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$

THE VECTOR $\vec{V}$ REPRESENTED BY $\overrightarrow{AB}$ (VECTOR FROM A TO B)

IS GIVEN BY

$$\vec{V} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$



e.g. FIND THE VECTOR REPRESENTED BY THE DIRECTED LINE SEGMENT FROM $(3, -7, 5)$ TO $(5, 3, -6)$

$$\langle 5-3, 3-(-7), -6-5 \rangle$$

$$\langle 2, 10, -11 \rangle$$

DEF:

THE MAGNITUDE (LENGTH) OF A VECTOR $\vec{v} = \langle x, y \rangle$ OR $\vec{v} = \langle x, y, z \rangle$ IS DENOTED $|\vec{v}|$ OR $\|\vec{v}\|$ AND IS GIVEN BY

(DISTANCE FROM (x, y, z) TO ORIGIN)

$$|\vec{v}| = \sqrt{x^2 + y^2 + z^2} \qquad (3D)$$

$$|\vec{v}| = \sqrt{x^2 + y^2} \qquad (2D)$$

0 SCALAR

NOTE: $|\vec{v}| \geq 0$ AND $|\vec{v}| = 0$ IF AND ONLY IF $\vec{v} = \underbrace{\vec{0} = \langle 0, 0, 0 \rangle}_{\text{DEF.}}$

ADDITION: $\vec{a} = \langle x_1, y_1, z_1 \rangle$, $\vec{b} = \langle x_2, y_2, z_2 \rangle$

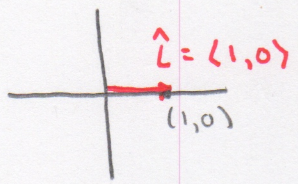
$$\vec{a} \pm \vec{b} = \langle x_1 \pm x_2, y_1 \pm y_2, z_1 \pm z_2 \rangle$$

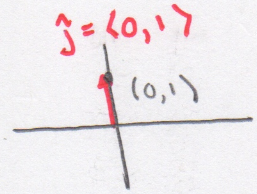
$\vec{a} - \vec{b}$

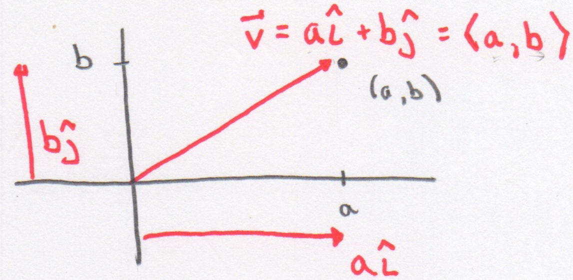
SCALAR MULTIPLICATION: $c\vec{a} = c\langle x_1, y_1, z_1 \rangle = \langle cx_1, cy_1, cz_1 \rangle$

STANDARD BASIS VECTORS

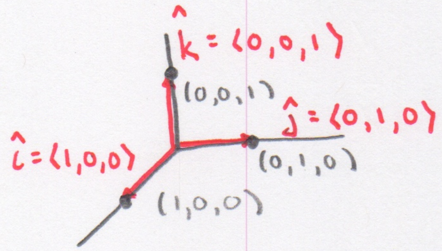
2D :







3D:



$\hat{i}, \hat{j}, \hat{k}$ ARE VECTORS WITH LENGTH 1 (i.e. UNIT VECTORS)

$$\vec{v} = \langle 3, -4, 2 \rangle = 3\hat{i} - 4\hat{j} + 2\hat{k}$$

$$3\underbrace{\langle 1,0,0 \rangle}_{\hat{i}} - 4\underbrace{\langle 0,1,0 \rangle}_{\hat{j}} + 2\underbrace{\langle 0,0,1 \rangle}_{\hat{k}}$$

$\vec{V}$

"i - HAT"

"j - HAT"

e.g. $\vec{a} = 3\hat{i} + 7j - \hat{k}$, $\vec{b} = 7j - 6k + 4i$

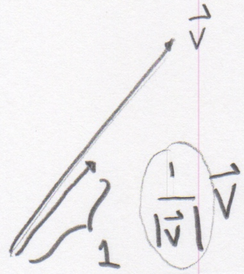
$$4\hat{i} + 7\hat{j} - 6\hat{k}$$

$$\vec{a} + \vec{b} = \langle 3, 7, -1 \rangle + \langle 4, 7, -6 \rangle$$

$$= \langle 3+4, 7+7, -1-6 \rangle$$

$$= \langle 7, 14, -7 \rangle = 7\langle 1, 2, -1 \rangle$$

THE UNIT VECTOR THAT POINTS IN THE SAME DIRECTION AS A GIVEN (NON-ZERO) VECTOR $\vec{v}$ IS $\boxed{\dfrac{1}{|\vec{v}|}\vec{v}}$



i.e. $\left| \dfrac{1}{|\vec{v}|}\vec{v} \right| = 1$

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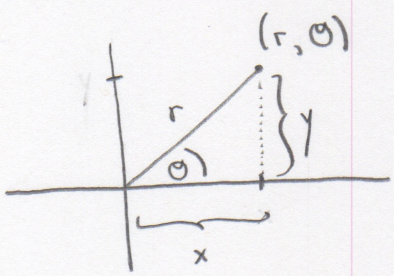
e.g. FIND UNIT VECTOR IN SAME DIRECTION AS $\boxed{\langle -1, 12, -12 \rangle}$

$$\frac{1}{\sqrt{\underbrace{(-1)^2}_{1} + \underbrace{(12)^2}_{144} + \underbrace{(-12)^2}_{144}}} \langle -1, 12, -12 \rangle = \frac{1}{17}\langle -1, 12, -12 \rangle$$

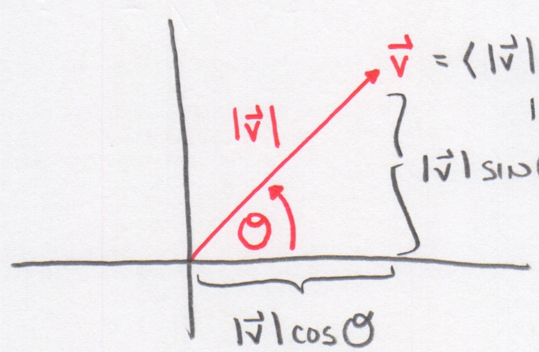
$$= \left\langle \frac{-1}{17}, \frac{12}{17}, \frac{-12}{17} \right\rangle$$

(2D) RECALL FROM POLAR COORD:

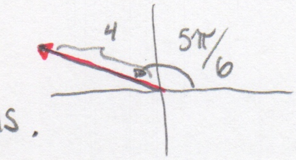


$x = r\cos\theta$

$y = r\sin\theta$



$\vec{v} = \langle |\vec{v}|\cos\theta, |\vec{v}|\sin\theta \rangle$

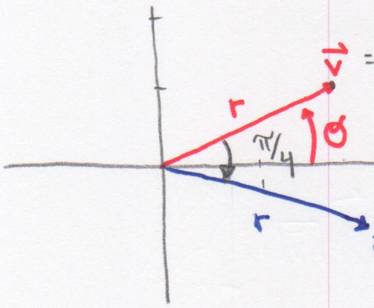


e.g. A VECTOR OF LENGTH 4 MAKES ANGLE OF $\frac{5\pi}{6}$ WITH POSITIVE X-AXIS. FIND COMPONENTS OF THE VECTOR. $\langle 4\cos\frac{5\pi}{6}, 4\sin\frac{5\pi}{6} \rangle$

e.g. ROTATE THE VECTOR $\vec{v} = \langle \sqrt{3}, 1 \rangle$ BY $\frac{\pi}{4}$ RADIANS CLOCKWISE TO FORM $\vec{w}$.

FIND COMPONENTS OF $\vec{w}$.

$$r = |\vec{v}| = \sqrt{\sqrt{3}^2 + 1^2} = 2$$



$$\vec{v} = \langle \sqrt{3}, 1 \rangle = \langle r\cos\theta, r\sin\theta \rangle$$

$$\sqrt{3} = 2\cos\theta \rightarrow \cos\theta = \frac{\sqrt{3}}{2}$$

$$1 = 2\sin\theta \rightarrow \sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}$$

$$\vec{w} = \left\langle r\cos\left(\theta - \frac{\pi}{4}\right), r\sin\left(\theta - \frac{\pi}{4}\right) \right\rangle$$

$$= \boxed{\left\langle 2\cos\left(\frac{\pi}{6} - \frac{\pi}{4}\right), 2\sin\left(\frac{\pi}{6} - \frac{\pi}{4}\right) \right\rangle}$$