§10.8 TAYLOR & MACLAURIN SERIES

RECALL FROM §10.7

ex 1: $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots$

CONVERGES TO $\frac{1}{1-x}$ IF $|x| < 1$.

ex 7: $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

CONVERGES TO $\tan^{-1} x$ IF $|x| \leq 1$

SO THESE FUNCTIONS HAVE POWER SERIES REPRESENTATIONS ON THOSE INTERVALS.

WE KNOW THAT WITHIN ITS INTERVAL OF CONVERGENCE, THE SUM OF A POWER SERIES IS A FUNCTION OF X WITH DERIVATIVES OF ALL ORDERS.

QUESTION: IF A FUNCTION $f(x)$ HAS DERIVATIVES OF ALL ORDERS ON AN INTERVAL, DOES IT HAVE A POWER SERIES REPRESENTATION ON AT LEAST PART OF THAT INTERVAL?

ASSUME $f(x)$ HAS DERIVATIVES OF ALL ORDERS ON AN INTERVAL CONTAINING $x = a$.

ASSUME $f(x)$ HAS A POWER SERIES REPRESENTATION ABOUT $x = a$, WITH POSITIVE RADIUS OF CONVERGENCE.

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots$$

THEN $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \ldots$

$$f''(x) = 2c_2 + 3\cdot 2\, c_3(x-a) + 4\cdot 3\cdot c_4 (x-a)^2 + \ldots$$

$$f'''(x) = 3\cdot 2\, c_3 + 4\cdot 3\cdot 2\, c_4\cdot(x-a) + \ldots$$

$$\vdots$$

$$f^{(n)}(x) = n!\, c_n + (x-a)\left[\quad \cdots\cdots \quad\right]$$

IF WE SET $x = a$,

$$f^{(n)}(a) = n!\, c_n \implies \boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

$\therefore$ IF f HAS A POWER SERIES REPRESENTATION AT $x = a$,

IT MUST BE

$$\boxed{\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots \end{aligned}}$$

<u>Def</u>: THIS IS CALLED THE <u>TAYLOR SERIES GENERATED BY f AT $x = a$</u>. IF $a = 0$, IT IS ALSO CALLED THE <u>MACLAURIN SERIES OF f</u>.

<u>Def</u>: $$f(x) \approx \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

IS THE <u>TAYLOR POLYNOMIAL OF ORDER N</u> GENERATED BY f AT $x = a$.

(BEST POLYNOMIAL APPROX. TO f WITH DEGREE $\le N$ IN A NEIGHBORHOOD OF a.)

ex 1. FIND TAYLOR SERIES

FOR $f(x) = \sin x$

AT $x = 0$

(MACLAURIN SERIES)

+ TAYLOR POLYNOMIAL OF ORDER 6.

ex 2. FIND TAYLOR SERIES FOR $f(x) = \frac{2+x}{1-x}$

AT $x = 0$

(MACLAURIN SERIES)

ex 3. FIND TAYLOR SERIES

FOR $f(x) = 2x^3 + x^2 + 3x - 8$

AT $x = 1$.

ex 4. FIND THE TAYLOR SERIES FOR $f(x) = 2^x$

AT $x = 1$.

ex 1. FIND TAYLOR SERIES FOR $f(x) = \sin x$ AT $x = 0$ (a).

(MACLAURIN SERIES)

+ TAYLOR POLYNOMIAL OF ORDER 6.

$$f(x) = \sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(a)}{n!}}_{c_n} (x-a)^n$$

$a = 0$

$$\begin{aligned} f(x) &= \sin(x) & f^{(0)}(0) &= 0 \\ f'(x) &= \cos(x) & f'(0) &= 1 \\ f''(x) &= -\sin(x) & f''(0) &= 0 \\ f'''(x) &= -\cos(x) & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin(x) & f^{(4)}(0) &= 0 \\ &\vdots & &\vdots \end{aligned}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \underbrace{\frac{0}{0!}x^0} + \frac{1}{1!}x^1 + \underbrace{\frac{0}{2!}x^2} + \frac{-1}{3!}x^3$$

$$= \boxed{\boxed{x^1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5} - \frac{1}{7!}x^7 + \dots}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$