§10.5 ABSOLUTE CONVERGENCE; THE RATIO & ROOT TESTS

Def: $\sum a_n$ CONVERGES ABSOLUTELY IF $\sum |a_n|$ CONVERGES.

e.g. $\frac{2}{5} - \frac{4}{25} + \frac{8}{125} - \frac{16}{625} + \dots$ $\qquad \sum \frac{2}{5}\left(-\frac{2}{5}\right)^{n-1} = \frac{2/5}{1+2/5} = \frac{2}{7}$

CONVERGES: $a_n = \frac{2}{5}\left(-\frac{2}{5}\right)^{n-1}$ GEOMETRIC WITH ~~$|r|<1$~~ ✓ $r = -\frac{2}{5}$

Note: $|a_n| = \frac{2}{5}\left(\frac{2}{5}\right)^{n-1}$

$$\sum |a_n| = \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \frac{16}{625} + \dots = \sum \frac{2}{5}\left(\frac{2}{5}\right)^n = \frac{2/5}{1-2/5} = \frac{2}{3}$$

Geo w/ $r = \frac{2}{5}$

$\therefore$ SERIES CONVERGES ABSOLUTELY.

THM: (ABSOLUTE CONVERGENCE TEST)

IF $\sum_{n=1}^{\infty} |a_n|$ CONVERGES THEN $\sum_{n=1}^{\infty} a_n$ CONVERGES

PROOF:

$\forall n, \quad -|a_n| \leq a \leq |a_n| \quad \Rightarrow \quad 0 \leq a + |a_n| \leq \underbrace{\underbrace{2|a_n|}_{\sum \text{ CONV.}}}_{\sum \text{ ALSO CONV.}}$

$\sum a + |a_n|$ CONVERGES BY D.C.T.

THEN $\sum a_n = \sum \left(a_n + |a_n| - |a_n|\right) = \underbrace{\underbrace{\sum \left(a_n + |a_n|\right)}_{\checkmark} - \underbrace{\sum |a_n|}_{\checkmark}}_{\text{CONVERGES } \checkmark}$

e.g. $\sum \frac{\sin(n)}{n\sqrt{n}}$ CONVERGES BECAUSE IT CONVERGES ABSOLUTELY BY COMP. TEST.

THE RATIO TEST LET $\sum a_n$ BE ANY SERIES AND SUPPOSE

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$$

THEN $\sum a_n$ $\begin{cases} \text{CONVERGES ABSOLUTELY IF } \rho < 1 \\ \text{DIVERGES IF } \rho > 1 \\ \text{INCONCLUSIVE IF } \rho = 1. \end{cases}$

IF $\rho > 1$ THEN TERMS DO NOT GO TO 0. $\Rightarrow$ DIVERGE BY n^{TH}-TERM TEST.

PROOF: IDEA: FOR LARGE n, SAY $n \geq N$, WE HAVE

$$\left| \frac{a_{n+1}}{a_n} \right| \approx \rho \quad \Rightarrow \quad \underbrace{|a_{n+1}| \approx \rho |a_n|}_{\approx \text{ GEOMETRIC.}}$$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n| \approx |a_N| + \rho|a_N| + \rho^2 |a_N| + \dots$$

$$\approx \underbrace{\sum_{n=1}^{N-1} |a_n|}_{\text{FINITE}} + \underbrace{\sum_{n=1}^{\infty} |a_N| \rho^{n-1}}_{\text{CONVERGES IF } \rho < 1}$$

$\rho = 1$ FOR BOTH $\sum \frac{1}{n}$ & $\sum \frac{1}{n^2}$

$\downarrow$ DIV. $\quad \downarrow$ CONV.

NOTE: RATIO TEST IS <u>GREAT</u> WHEN a_n IS AN EXPRESSION WITH $n!$

ex. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{3^n}$

ex. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2 (n+2)!}{n!\, 3^{2n}}$

THE ROOT TEST: LET $\sum a_n$ BE ANY SERIES AND SUPPOSE THAT

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$$

THEN $\sum a_n$ $\begin{cases} \text{CONVERGES ABSOLUTELY IF } \rho < 1 \\ \text{DIVERGES IF } \rho > 1 \\ \text{INCONCLUSIVE IF } \rho = 1. \end{cases}$

PROOF: FOR n LARGE ENOUGH, SAY $n \geq N$, WE HAVE

$$\sqrt[n]{|a_n|} \approx \rho \quad \Rightarrow \quad |a_n| \approx \rho^n$$

NOTE: IF $\rho > 1$ THEN $a_n \not\to 0$ & $\sum a_n$ DIVERGES.

$$\therefore \sum_{n=1}^{\infty} |a_n| = \underbrace{\sum_{n=1}^{N-1} |a_n|}_{\text{FINITE}} + \underbrace{\sum_{n=N}^{\infty} \rho \cdot \rho^{n-1}}_{\text{CONVERGES IF } \rho < 1}$$

FOR $\rho = 1$ CONSIDER $\underbrace{\sum \frac{1}{n}}_{\text{DIV.}}$ & $\underbrace{\sum \frac{1}{n^2}}_{\text{CONV.}}$: $\displaystyle\lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = 1$

ex. $\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$

ex. $\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5} \right)^n$

ex. $\sum_{n=1}^{\infty} \left[-\ln\left(e^2 + \frac{1}{n}\right) \right]^{n+1} = \sum_{n=0}^{\infty} \left[-\ln\left(e^2 + \frac{1}{n+1}\right) \right]^n$

ex. $\sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{1}{n}\right)^{n^2}$ HINT: $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

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ex. $\sum_{n=1}^{\infty} \frac{n! \ln n}{n(n+2)!}$

$\sum \operatorname{EXP}\left[n^2 \ln 2 - 2^n \ln n \right]$

ex. $\sum_{n=3}^{\infty} \frac{2^{n^2}}{n^{2^n}}$

$$\frac{\dfrac{2^{(n+1)^2}}{(n+1)^{2^{(n+1)}}}}{\dfrac{2^{n^2}}{n^{2^n}}} = \frac{2^{\cancel{n^2}+2n+1}}{\cancel{2^{n^2}}} \cdot \frac{n^{2^n}}{(n+1)^{2^{n+1}}}$$

$$= \frac{2^{2n+1}\, n^{2^n}}{(n+1)(n+1)^{2^n}} = \frac{2^{2n+1}}{n+1} \left(\frac{n}{n+1} \right)^{2^n}$$

$= \infty \cdot 1^{\infty}$

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$$\cancel{\lim_{n\to\infty} \left(\frac{2^{n^2}}{n^{2^n}} \right)^{1/n} = \lim_{n\to\infty} \frac{2^n}{n^{2^n/n}}}$$

$$\operatorname{EXP}\left[\lim_{n\to\infty} \frac{2^n \ln\left(\frac{n}{n+1}\right)}{2^{-n}} \right] = \operatorname{EXP}\left[\lim_{n\to\infty} \frac{\frac{n+1}{n}}{-\ln 2 \cdot \cancel{2^{-n}}} \right]$$

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