§ 10.4 COMPARISON TESTS

THM (DIRECT COMPARISON TEST)

LET $\sum a_n$ & $\sum b_n$ BE TWO SERIES S.T. $0 \le a_n \le b_n \ \forall n$.

THEN 1. IF $\sum b_n$ CONVERGES THEN $\sum a_n$ CONVERGES.

2. IF $\sum a_n$ DIVERGES THEN $\sum b_n$ DIVERGES.

PROOF: TERMS ALL NON-NEG. SO SEQ. OF PARTIAL SUMS IS NON-DECREASING.
$\therefore$ SEQ. OF PARTIAL SUMS (SERIES) CONVERGES IFF BOUNDED FROM ABOVE.

1: FOR ANY n, WE HAVE

$$a_1 + a_2 + \dots + a_n \le b_1 + b_2 + \dots + b_n \le \sum_{n=1}^{\infty} b_n = M$$

PARTIAL SUMS ALL BOUNDED ABOVE BY M.

2: $$\underbrace{a_1 + a_2 + \dots + a_n}_{\text{UNBOUNDED AS } n \text{ INCR.}} \le \underbrace{b_1 + b_2 + \dots + b_n}_{\text{MUST ALSO BE UNBOUNDED AS } n \text{ INCR.}}$$

ex. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$

ex. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n 3^n}$

ex. $$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \le 1 + \underbrace{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots}_{\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 2}$$

THM (LIMIT COMPARISON TEST)

SUPPOSE $a_n > 0$ & $b_n > 0$ $\forall n \geq N$

1. IF $\lim_{n \to \infty} \frac{a_n}{b_n} = c$ AND $0 < c < \infty$, THEN $\sum a_n$ & $\sum b_n$

   EITHER BOTH CONVERGE OR BOTH DIVERGE.

2. IF $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ AND $\sum b_n$ CONVERGES THEN $\sum a_n$ CONVERGES.

3. IF $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ AND $\sum b_n$ DIVERGES THEN $\sum a_n$ DIVERGES.

PROOF: 1. SINCE $\lim_{n \to \infty} \frac{a_n}{b_n} = c$, $\exists N$ S.T. WHEN $n \geq N$ WE HAVE

$$\left| \frac{a_n}{b_n} - c \right| < \frac{c}{2}$$ ANY POSITIVE # WILL DO

AND SO $$-\frac{c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2}$$

$$\frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2}$$

$$\left(\frac{c}{2}\right) b_n < a_n < \left(\frac{3c}{2}\right) b_n$$

IF $\sum b_n$ DIVERGES

THEN $\sum \frac{c}{2} b_n$ DIVERGES

$\Rightarrow \sum a_n$ DIVERGES. DCT.

IF $\sum b_n$ CONVERGES

THEN $\sum \frac{3c}{2} b_n$ CONVERGES

$\Rightarrow \sum a_n$ CONVERGES DCT

ex. $\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$

$y = \ln(1+w) \approx w$

FOR SMALL w

ex. $\sum \ln\left(1+\frac{1}{n^2}\right)$ LIMIT COMP. w/ $\frac{1}{n^2}$

TANGENT LINE

↓

$\left(y = \ln x \approx x - 1 \quad \text{FOR } x \text{ NEAR } 1 \right)$

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n^2}} \xrightarrow{L'HO} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n^2}}\left(-2n^{-3}\right)}{-2n^{-3}}$$

$$= \cancel{(-1)} \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1$$

n	2^n
1	2
2	4
3	8

a_n b_n

ex. $\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n}$ COMPARE WITH $\frac{1}{n^2}$ $\quad \frac{2\cdot \cancel{2^n}}{n^2\cdot \cancel{2^n}} > \frac{n+2^n}{n^2 2^n} > \frac{n}{n^2 2^n} = \frac{1}{n2^n}$ ~~∞~~ SERIES CONV.

$$\lim_{n\to\infty} \frac{n+2^n}{\cancel{n^2}\, 2^n}\cdot\frac{\cancel{n^2}}{1} = \lim_{n\to\infty} \frac{n+2^n}{2^n} \overset{L'HO}{=} \lim_{n\to\infty} \frac{1+2^n \ln 2}{2^n \ln 2} = 1$$

ex. $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n} = a_n$ LET $b_n = \left(\frac{3}{4}\right)^n$ GEOMETRIC

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{2^n+3^n}{3^n+4^n}\cdot\left(\frac{4}{3}\right)^n = \lim_{n\to\infty} \frac{\left(\frac{2}{3}\right)^n + 1}{}$$

$$= \lim_{n\to\infty} \frac{8^n+12^n}{9^n+12^n} = \lim_{n\to\infty} \frac{\left(\frac{8}{12}\right)^n+1}{\left(\frac{9}{12}\right)^n+1} = 1$$