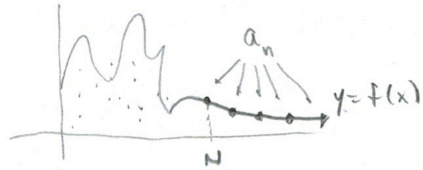
§ 10.3 THE INTEGRAL TEST



THM 9. $\{a_n\}$ SEQUENCE OF POSITIVE TERMS.

RELATED CONTINUOUS FUNCTION $f(n) = a_n$, $\forall n \geq N$

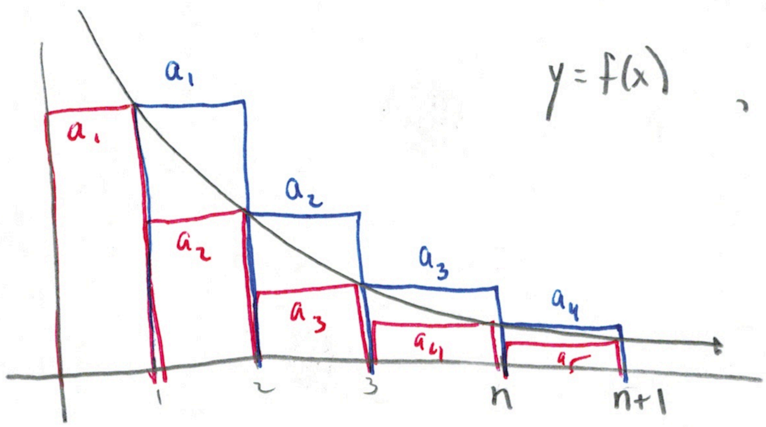
$f(x)$ IS POSITIVE & DECREASING FOR ALL $x \geq N$ (SOME INTEGER)

THEN THE SERIES $\sum_{n=N}^{\infty} a_n$ AND THE INTEGRAL $\int_N^{\infty} f(x)\,dx$ EITHER

BOTH CONVERGE OR BOTH DIVERGE.

PROOF

(ASSUME N = 1)



$y = f(x)$, $f(n) = a_n$

$$\int_1^{n+1} f(x)\,dx < a_1 + a_2 + \dots + a_n$$

$$a_2 + a_3 + \dots + a_n < \int_1^{n} f(x)\,dx$$

$+a_1$ $\qquad\qquad$ $+a_1$

$$\int_1^{n+1} f(x)\,dx < a_1 + a_2 + a_3 + \dots + a_n < a_1 + \int_1^{n} f(x)\,dx$$

TRUE FOR ALL n,

TRUE AS $n \to \infty$.

ex. p - SERIES $\sum_{n=1}^{\infty} \frac{1}{n^p}$ CONVERGES IF $p > 1$

DIVERGES IF $p \le 1$.

(DOES ~~NOT~~ CONVERGE TO $\frac{1}{p-1}$)

ex. $\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$

$$f(x) = x^2 e^{-x/3}$$

$$f'(x) = 2x e^{-x/3} - \frac{1}{3} x^2 e^{-x/3}$$

$$= \left(2x - \frac{1}{3}x^2\right) e^{-x/3}$$

$$= x\left(2 - \frac{1}{3}x\right) e^{-x/3}$$

	$x<0$	$0<x<6$	$x>6$
x	$-$	$+$	$+$
$2-\frac{1}{3}x$	$+$	$+$	$-$

$\therefore$ f IS DECREASING FOR $x \ge 6$

$$\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}} = \underbrace{\sum_{n=1}^{5} \frac{n^2}{e^{n/3}}}_{\text{FINITE}} + \sum_{n=6}^{\infty} \frac{n^2}{e^{n/3}}$$

FINITE

CONVERGES

$\updownarrow$ SAME BEHAVIOR AS

$$\int_6^{\infty} x^2 e^{-x/3}\, dx$$

$u = x^2$ $v = -3e^{-x/3}$

$du = 2x\,dx$ $dv = e^{-x/3}\,dx$

$u = x$ $v = -3e^{-x/3}$

$du = dx$ $dv = e^{-x/3}\,dx$

$$= -3x^2 e^{-x/3} + 6\int x e^{-x/3}\, dx$$

$$= -3x^2 e^{-x/3} + 6\left[-3x e^{-x/3} + 3\int e^{-x/3}\, dx\right]$$

$$= -3x^2 e^{-x/3} - 18x e^{-x/3} - 54 e^{-x/3} = -3e^{-x/3}\left(x^2 + 6x + 18\right)$$

ex. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n^2)}{n}$

$$f(x) = \frac{\ln(x^2)}{x} = \frac{2\ln x}{x}$$

$$f'(x) = 2\left(\frac{x \cdot \frac{1}{x} - \ln x}{x^2}\right) = \frac{2(1-\ln x)}{x^2}$$

NEG WHEN $1 - \ln x < 0 \rightarrow 1 < \ln x$

$$\underline{\underline{e < x}}$$

$$\sum_{n=2}^{\infty} \frac{\ln(n^2)}{n} = \frac{\ln(4)}{2} + \sum_{n=3}^{\infty} \frac{\ln(n^2)}{n}$$

$$\uparrow$$

$$\int_3^{\infty} \frac{\ln(x^2)}{x}\,dx = 2\int_3^{\infty} \frac{\ln x}{x}\,dx \qquad u = \ln x \quad du = \frac{1}{x}\,dx$$

$$= 2\int_{\ln 3}^{\infty} u\,du = 2\left[\lim_{t\to\infty} \frac{1}{2}u^2 \Big|_{\ln 3}^{t}\right]$$

$$= \frac{\cancel{2}}{\cancel{2}}\left[\lim_{t\to\infty} t^2 - (\ln 3)^2\right] = \infty$$