Name: *Answer Key*

Math 205 Elements of Calculus

11/16/2017

Practice Exam 2

1. Find the derivative of the following functions.

(a) $F(x) = (4x^5 + \sqrt{x})(5x^2 + 2x + 4 + \frac{1}{x})$ (PRODUCT RULE)

$$F'(x) = \frac{d}{dx}\left[4x^5 + x^{1/2}\right](5x^2 + 2x + 4 + x^{-1}) + (4x^5 + x^{1/2})\frac{d}{dx}\left[5x^2 + 2x + 4 + x^{-1}\right]$$

$$\boxed{F'(x) = \left(20x^4 + \frac{1}{2}x^{-\frac{1}{2}}\right)(5x^2 + 2x + 4 + x^{-1}) + (4x^5 + x^{1/2})(10x + 2 - x^{-2})}$$

(b) $G(x) = \dfrac{2\sqrt[3]{x} + 3x^4}{4x + 3}$ (QUOTIENT RULE)

$$G'(x) = \frac{\frac{d}{dx}\left[2x^{1/3} + 3x^4\right](4x+3) - (2x^{1/3} + 3x^4)\frac{d}{dx}\left[4x + 3\right]}{(4x+3)^2}$$

$$\boxed{G'(x) = \frac{(\frac{2}{3}x^{-2/3} + 12x^3)(4x+3) - (2x^{1/3} + 3x^4)(4)}{(4x+3)^2}}$$

(c) $H(x) = e^x - 2^x$

$$\boxed{H'(x) = e^x - 2^x \ln(2)}$$

(d) $I(x) = \ln x$

$$\boxed{I'(x) = \frac{1}{x}}$$

2. Find the derivative of the following functions. (CHAIN RULE)

(a) $P(x) = (2x^4 - 8x^2 + 1)^8$

$$\boxed{P'(x) = 8(2x^4 - 8x^2 + 1)^7 (8x^3 - 16x)}$$

(b) $Q(x) = \sqrt{3x^2 + x}$

$$\boxed{Q'(x) = \frac{1}{2}(3x^2 + x)^{-1/2} (6x + 1)}$$

(c) $R(x) = \ln(4x^2 + 5x^{-2})$

$$\boxed{R'(x) = \frac{1}{4x^2 + 5x^{-2}} (8x - 10x^{-3})}$$

(d) $S(x) = e^{2x^2+6x-1} + 5^{4x^3}$

$$\boxed{S'(x) = e^{2x^2 + 6x - 1} (4x + 6) + 5^{4x^3} \cdot \text{Ln}(5)(12x^2)}$$

3. For each of the following equations, find $\frac{dy}{dx}$. (IMPLICIT DIFFERENTIATION)

(a) $x^2 - y^6 = 2$

$$\frac{d}{dx}\left[x^2 - y^6\right] = \frac{d}{dx}\left[2\right] \Rightarrow 2x - 6y^5\frac{dy}{dx} = 0$$

$$\Rightarrow -6y^5\frac{dy}{dx} = -2x \Rightarrow \boxed{\frac{dy}{dx} = \frac{x}{3y^5}}$$

(b) $x^4 + \boxed{x^2y^3} - y = 7$

$$4x^3 + \boxed{2xy^3 + x^2 3y^2\frac{dy}{dx}} - \frac{dy}{dx} = 0$$

$$3x^2y^2\frac{dy}{dx} - \frac{dy}{dx} = -4x^3 - 2xy^3$$

$$(3x^2y^2 - 1)\frac{dy}{dx} = -4x^3 - 2xy^3 \Rightarrow \boxed{\frac{dy}{dx} = \frac{-4x^3 - 2xy^3}{3x^2y^2 - 1}}$$

4. Give an equation for the tangent line to the curve

$$e^{xy} = x^2 + y^2$$

at the point $(0, 1)$.

IMPLICIT DIFFERENTIATION: $e^{xy}\left(y + x\frac{dy}{dx}\right) = 2x + 2y\frac{dy}{dx}$

$$e^{xy}y + e^{xy}x\frac{dy}{dx} = 2x + 2y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x - e^{xy}y}{e^{xy}x - 2y} \qquad \therefore \frac{dy}{dx} \text{ AT } (0,1) = \frac{2(0) - e^{(0)(1)}(1)}{e^{(0)(1)}(0) - 2(1)} = \frac{1}{2}$$

$\therefore$ TANGENT LINE: $\boxed{y - 1 = \frac{1}{2}(x - 0)}$

5. Suppose you deposit \$1400 into a savings account with an annual interest rate of 4.5% that is compounded quarterly.

(a) How much will your savings be worth after 3 years?

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A(t) = 1400\left(1 + \frac{.045}{4}\right)^{4t}$$

$$A(3) = 1400\left(1 + \frac{.045}{4}\right)^{4\cdot 3} \approx \boxed{\$1601.14}$$

(b) How long will it take you savings to double? (\$2800)

SOLVE FOR t: $A(t) = 1400\left(1 + \frac{.045}{4}\right)^{4t} = 2800$

$$\left(1 + \frac{.045}{4}\right)^{4t} = 2 \quad \Rightarrow \quad 4t \ln\left(1 + \frac{.045}{4}\right) = \ln(2)$$

$$t = \frac{\ln(2)}{4\ln\left(1 + \frac{.045}{4}\right)} \approx \boxed{15.49 \text{ YEARS}}$$

6. Suppose a population of bacteria doubles every 2 hours. If the population at 5pm is 2400, what was the population at noon (5 hours earlier)?

LET t = # HOURS AFTER NOON, P_0 = POPULATION AT $t = 0$ (NOON)

$P(t) = P_0 e^{kt}$. DOUBLES EVERY HOUR $\Rightarrow P(2) = P_0 e^{2k} = 2P_0$

$$\Rightarrow e^{2k} = 2 \Rightarrow 2k = \ln(2) \Rightarrow k = \ln(2)/2$$

$$\therefore \quad P(t) = P_0\left[e^{\ln(2)/2}\right]^t = P_0 \cdot 2^{t/2}$$

$$P(5) = P_0 \cdot 2^{5/2} = 2400 \quad \Rightarrow \quad P_0 = \frac{2400}{2^{5/2}} \approx \boxed{424}$$

7. Suppose a sample of radioactive material is observed to decay to 72% of its original mass after 18 years. Find the half-life of this material.

$$M(t) = M_0 e^{kt}$$

$$M(18) = M_0 e^{18k} = .72\, M_0 \Rightarrow e^{18k} = .72 \Rightarrow k = \frac{\ln(.72)}{18}$$

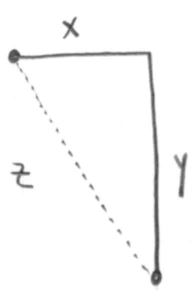
$$\therefore\ M(t) = M_0\, e^{\ln(.72)\cdot \frac{t}{18}} = M_0\,(.72)^{t/18}$$

HALF-LIFE: SOLVE FOR t: $M(t) = M_0(.72)^{t/18} = \frac{1}{2} M_0$

$$(.72)^{t/18} = \frac{1}{2} \Rightarrow \frac{t}{18}\ln(.72) = \ln\left(\frac{1}{2}\right)$$

$$t = \frac{18\ln\left(\frac{1}{2}\right)}{\ln(.72)} \approx \boxed{37.98 \text{ YEARS}}$$

8. Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?



$$\frac{dx}{dt} = 25, \quad \frac{dy}{dt} = 60$$

AFTER 2 HOURS,

$$x = (2)(25) = 50$$

$$y = (2)(60) = 120$$

$$z = \sqrt{50^2 + 120^2} = 130$$

$$z^2 = x^2 + y^2$$

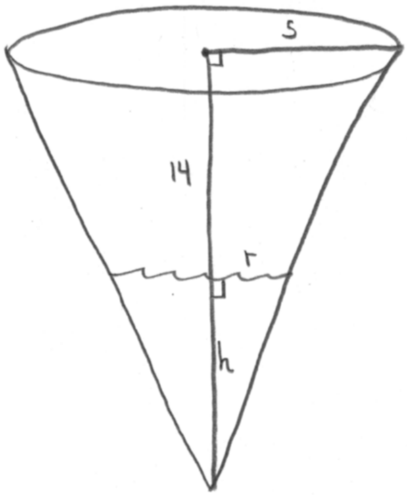
$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{x\cdot\frac{dx}{dt} + y\cdot\frac{dy}{dt}}{z}$$

$$\left.\frac{dz}{dt}\right|_{t=2} = \frac{50(25) + 120(60)}{130}$$

$$= \boxed{65 \text{ mi/H}}$$

9. A water tank has the shape of an inverted circular cone (the "tip" of the cone is at the bottom) with a base radius of 5 ft and a height of 14 ft. At what rate is the depth of the water in the tank changing when the depth of the water is 6 ft?



SIMILAR TRIANGLES: $\frac{r}{h} = \frac{5}{14} \Rightarrow r = \frac{5}{14}h$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{5}{14}h\right)^2 h$$

$$V = \frac{25\pi}{3\cdot 196} h^3$$

$$\frac{dV}{dt} = \frac{25\pi}{196} h^2 \cdot \frac{dh}{dt}$$

$\frac{dV}{dt} = 25 \text{ ft}^3/\text{MIN}$

$$\therefore \quad \frac{dh}{dt} = \frac{196 \frac{dV}{dt}}{25\pi h^2}$$

NOW PLUG IN $h = 6$

$\frac{dV}{dt} = 25$

$$\frac{dh}{dt} = \boxed{\frac{196}{36\pi} \approx 17.1 \text{ ft/MIN}}$$

10. Find the critical numbers for the following functions.

(a) $f(x) = x \ln x$

$f'(x) = 1 \ln x + x \cdot \frac{1}{x} = \ln x + 1$

$f'(x)$ IS UNDEFINED FOR $x \leq 0$, BUT NEITHER IS f (DOM(f) : $x > 0$)

$f'(x) = 0$ WHEN $\ln x = -1$

$$\boxed{x = e^{-1} = \frac{1}{e}}$$

(b) $g(x) = \sqrt{1 - x^2}$

$g'(x) = \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{1-x^2}}$

$g'(x)$ UNDEFINED WHEN $\boxed{x = \pm 1}$ (ALSO WHEN $|x| > 1$, BUT THEN g IS ALSO UNDEFINED)

$g'(x) = 0$ WHEN $\boxed{x = 0}$

11. Find the absolute maximum and minimum values of

$$f(x) = \frac{x}{x^2 - x + 1}$$

over the closed interval $0 \leq x \leq 3$.

$$f'(x) = \frac{(1)(x^2 - x + 1) - x(2x - 1)}{(x^2 - x + 1)^2} = \frac{1 - x^2}{(x^2 - x + 1)^2}$$

$f'(x)$ IS ALWAYS DEFINED, AND $f'(x) = 0$ IF $x = \pm 1$ (CRITICAL POINTS)

CHECK VALUE OF f AT CRITICAL POINT(S) INSIDE DOMAIN $[0, 3]$

AND AT ENDPOINTS

x	$f(x)$
1	1
0	0
3	$\frac{3}{7}$

ABS. MAX VALUE 1

ABS. MIN VALUE 0

12. Let $f(x) = \ln(x^4 + 27)$.

(a) Find the intervals on which f is increasing/decreasing.

(b) List any/all local maximums and minimums.

(c) Find the intervals on which f is concave up/down.

(d) List any/all inflection points for the graph $y = f(x)$.

(e) Use the information from parts (a)-(d) to sketch (roughly) the graph $y = f(x)$.

(a) $f'(x) = \dfrac{4x^3}{x^4 + 27} = 0 \Rightarrow x = 0$

$(-\infty, 0)$ 0 $(0, \infty)$

$f'(x) < 0$ $f'(x) > 0$

∴ $\boxed{f \text{ DECREASING ON } (-\infty, 0), \quad f \text{ INCREASING ON } (0, \infty)}$

(b) $\boxed{x = 0 \text{ IS A LOCAL MINIMUM}}$ (1ST DERIVATIVE TEST)

(c) $f''(x) = \dfrac{12x^2(x^4 + 27) - 4x^3(4x^3)}{(x^4 + 27)^2} = 0$

$\Rightarrow -4x^6 + 324x^2 = 0 \Rightarrow -4x^4(x^2 - 81) = 0$

$-4x^2(x + 9)(x^2 - 9) = 0 \Rightarrow x = -3, 0, 3$

$(-\infty, -3)$ $(-3, 0)$ $(0, 3)$ $(3, \infty)$

-3 0 3

$f'' < 0$ $f'' > 0$ $f'' > 0$ $f'' < 0$

NOTE: CONCAVITY DOES NOT CHANGE AT $x = 0$

$$\boxed{\begin{array}{l} \text{CONCAVE UP ON } (-3, 0) \cup (0, 3) \text{ OR } (-3, 3) \\ \text{CONCAVE DOWN ON } (-\infty, -3) \cup (3, \infty) \end{array}}$$

(d) INFLECTION POINTS : $(-3, f(-3))$ & $(3, f(3))$, i.e. $\boxed{(\pm 3, \ln(108))}$

(e)