Name: *ANSWER KEY* (74 POINTS) 11/16/2017

Math 205 Elements of Calculus **Exam 2 Part I**

1. Find the derivative of the following functions.

4 (a) $F(x) = (3x^2 + 2x)^4 \ln(5x - \frac{1}{x})$

$$\boxed{F'(x) = 4(3x^2+2x)^3(6x+2)\ln\left(5x-\frac{1}{x}\right) + (3x^2+2x)^4 \frac{1}{5x-\frac{1}{x}}\left(5+\frac{1}{x^2}\right)}$$

4 (b) $G(x) = \dfrac{e^{7x^2-x}}{\sqrt{x^2+5}}$

$$\boxed{G'(x) = \frac{e^{7x^2-x}(14x-1)\sqrt{x^2+5} - e^{7x^2-x}\cdot\frac{1}{2}(x^2+5)^{-1/2}(2x)}{x^2+5}}$$

$14x^3 + 70x - x^2 - 5 - x$

$$\frac{e^{7x^2-x}(x^2+5)^{-\frac{1}{2}}\left[(14x-1)(x^2+5) - x\right]}{x^2+5} = \frac{e^{7x^2-x}(14x^3-x^2+69x-5)}{(x^2+5)^{5/2}}$$

4 (c) $H(x) = \sqrt[3]{\dfrac{x^2-3x}{x^4+6x}}$

$2x^5 + 12x^2 - 3x^4 - 18x \quad - (4x^5 + 6x^2 - 12x^4 - 18x)$

$$\boxed{H'(x) = \frac{1}{3}\left(\frac{x^2-3x}{x^4+6x}\right)^{-2/3} \cdot \frac{(2x-3)(x^4+6x) - (x^2-3x)(4x^3+6)}{(x^4+6x)^2}}$$

$$= \frac{1}{3}\left[\frac{\cancel{x}(x^3+6)}{\cancel{x}(x-3)}\right]^{2/3} \frac{-2x^5+9x^4+6x^2}{x^2(x^3+6)^2} = \frac{-2x^3+9x^2+6}{3(x-3)^{2/3}(x^3+6)^{4/3}}$$

8 2. Give an equation for the tangent line to the curve

$$2(x^2+y^2)^2 = 25(x^2-y^2)$$

at the point $(3,1)$.

$$4(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) = 25\left(2x-2y\frac{dy}{dx}\right)$$

$$8x^3 + 8x^2y\frac{dy}{dx} + 8xy^2 + 8y^3\frac{dy}{dx} = 50x - 50y\frac{dy}{dx}$$

$$8x^2y\frac{dy}{dx} + 8y^3\frac{dy}{dx} + 50y\frac{dy}{dx} = 50x - 8x^3 - 8xy^2$$

$$\frac{dy}{dx} = \frac{50x-8x^3-8xy^2}{8x^2y+8y^3+50y} \xrightarrow[(3,1)]{\text{AT}} = \frac{-90}{130}$$

$$\therefore \boxed{y-1 = -\frac{9}{13}(x-3) \quad \text{OR} \quad y = -\frac{9}{13}x + \frac{40}{13}}$$

3. Suppose you deposit \$600 into a savings account with an annual interest rate of 3.25% that is compounded monthly.

4 (a) How much will your savings be worth after 5 years?

$$A(t) = 600\left(1+\frac{.0325}{12}\right)^{12t}$$

$$A(5) = \boxed{600\left(1+\frac{.0325}{12}\right)^{60} \approx \boxed{\$705.71}}$$

4 (b) How long will it take your savings to double?

$$A(t) = 600\left(1+\frac{.0325}{12}\right)^{12t} = 1200$$

$$\left(1+\frac{.0325}{12}\right)^{12t} = 2$$

$$12t\ \ln\left(1+\frac{.0325}{12}\right) = \ln 2$$

$$\boxed{t = \frac{\ln 2}{12\ln\left(1+\frac{.0325}{12}\right)} \approx \boxed{21.36 \text{ YEARS}}}$$

4 4. Suppose a population of bacteria triples every 7 hours. If the population at 5pm is 3200, what was the population at noon (5 hours earlier)?

$P(t) = P_0 \cdot 3^{t/7}$ ← $P(7) = P_0 e^{7K} = 3P_0$

$e^{7K} = 3$

$K = \frac{\ln(3)}{7}$

$P(5) = P_0 \cdot 3^{5/7} = 3200$

$\boxed{P_0 = \frac{3200}{3^{5/7}} \approx \boxed{1460}}$

$\left(P(t) = P_0 e^{kt} \right)$

4 5. Suppose a sample of radioactive material is observed to decay to 84% of its original mass after 35 years. Find the half-life of this material.

$M(t) = M_0 (.84)^{t/35} = \frac{1}{2} M_0$

$\frac{t}{35} \ln(.84) = \ln\left(\frac{1}{2}\right)$

$t = \boxed{\frac{35 \ln\left(\frac{1}{2}\right)}{\ln(.84)} \approx 139.14 \text{ YEARS}}$

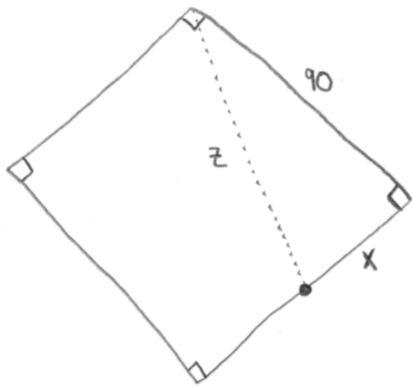
$M(35) = M_0 e^{35k} = .84 M_0$

$\left(M(t) = M_0 e^{kt} \right)$

$35k = \ln .84$

$k = \frac{1}{35} \ln .84$

8 6. A baseball diamond is a square with side 90 ft. A batter hits the ball and runs toward first base with a speed of 24 ft/s. At what rate is her distance from second base decreasing when she is halfway to first base?



$$x^2 + 90^2 = z^2$$

$$2x \frac{dx}{dt} = 2z \frac{dz}{dt}$$

$$\therefore \frac{dz}{dt} = \frac{x \frac{dx}{dt}}{z}$$

WHEN $x = 45$,

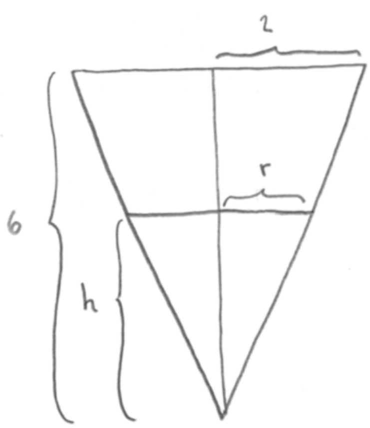
$$z = \sqrt{90^2 + 45^2} = 45\sqrt{5}$$

$$\frac{dx}{dt} = -24 \text{ ft/s}$$

PLUG IN

$$\frac{dz}{dt} = \frac{45(-24)}{45\sqrt{5}} = \boxed{\frac{-24}{\sqrt{5}} \text{ ft/s} \approx -10.73 \text{ ft/s}}$$

8 7. Water is being pumped into an inverted conical tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 0.2 m/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.



$$\frac{r}{h} = \frac{2}{6} \Rightarrow r = \frac{1}{3}h$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{1}{3}h\right)^2 h = \frac{\pi}{27} h^3$$

$$\frac{dV}{dt} = \frac{\pi}{9} h^2 \frac{dh}{dt}$$

PLUG IN $h = 2$, $\frac{dh}{dt} = .2$

$$\frac{dV}{dt} = \frac{\pi}{9}(2)^2(.2) = \boxed{\frac{4\pi}{45} \approx .279 \text{ m}^3/\text{MIN}}$$

4 8. Find the critical numbers for the function $f(x) = x^{4/5}(x-4)^2$.

$$f'(x) = \frac{4}{5}x^{-1/5}(x-4)^2 + x^{4/5} \cdot 2(x-4)$$

$$= x^{-1/5}(x-4)\left[\frac{4}{5}(x-4) + 2x\right] = \begin{cases} 0 & \text{OR} \\ \text{UNDEFINED} \end{cases}$$

UNDEFINED WHEN $X = 0$

$X = 4$

$$\frac{14}{5}x - \frac{16}{5} = 0$$

$$x = \frac{8}{7}$$

CRITICAL #'S

$$\boxed{x = 0, 4, \frac{8}{7}}$$

8 9. Find the absolute maximum and minimum values of

$$f(x) = x\sqrt{4-x^2}$$ DOMAIN $[-2, 2]$

over the closed interval $-1 \leq x \leq 2$.

CRITICAL #'S: $f'(x) = \sqrt{4-x^2} - \dfrac{x^2}{\sqrt{4-x^2}}$ (UNDEFINED WHEN $x = \pm 2$)

$f'(x) = 0 \longrightarrow \sqrt{4-x^2} = \dfrac{x^2}{\sqrt{4-x^2}}$

$$4 - x^2 = x^2$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

NOTE: $-\sqrt{2}$ IS NOT IN $[-1, 2]$

x	$f(x)$	
-1	$-\sqrt{3}$	ABS MIN VAL.
$\sqrt{2}$	2	ABS MAX VAL.
2	0	

10 10. Let $f(x) = (x^2 - 1)^3$.

(a) Find the intervals on which f is increasing/decreasing.

(b) List any/all local maximums and minimums.

(c) Find the intervals on which f is concave up/down.

(d) List any/all inflection points for the graph $y = f(x)$.

(e) Use the information from parts (a)-(d) to sketch (roughly) the graph $y = f(x)$.

(a) $f'(x) = 3(x^2-1)^2(2x)$

$= 6x(x+1)^2(x-1)^2 = 0$

$x = 0, \pm 1$

	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
f'	NEG	NEG	POS	POS
f	DECR	DECR	INCR	INCR

f IS INCREASING ON $(0,1) \cup (1,\infty)$

f IS DECREASING ON $(-\infty,-1) \cup (-1,0)$

(b) AND f HAS LOCAL MIN AT $x = 0$ WITH LOCAL MIN VALUE $f(0) = -1$

(c) $f''(x) = 6(x^2-1)^2 + 6x \cdot 2(x^2-1)(2x)$

$= 6(x^2-1)\left[(x^2-1) + 4x^2\right] = 0$

$6(x+1)(x-1)(5x^2-1) = 0$

$x = -1 \quad x = 1 \quad x = \pm\sqrt{\frac{1}{5}}$

	$x<-1$	$-1 < x < -\sqrt{\frac{1}{5}}$	$-\sqrt{\frac{1}{5}} < x < \sqrt{\frac{1}{5}}$	$\sqrt{\frac{1}{5}} < x < 1$	$x > 1$
f''	POS	NEG	POS	NEG	POS

f IS CONCAVE UP ON $(-\infty,-1) \cup (-\sqrt{1/5}, \sqrt{1/5}) \cup (1,\infty)$,

f IS CONCAVE DOWN ON $(-1, -\sqrt{1/5}) \cup (\sqrt{1/5}, 1)$,

(d) AN HAS POINTS OF INFLECTION AT $(-1, 0), (-\sqrt{1/5}, \approx -.512)$, $(\sqrt{1/5}, \approx -.512), (1, 0)$

(e)