Name: *ANSWER KEY* 11/6/2014

Math 203 Calculus III Midterm Exam

Please show all work and **box your final answers**. If you need more room, you may use the backs of the pages. Calculators are not allowed. Good luck!

1. Suppose the position of a particle at time t is given by the vector valued function

$$\vec{r}(t) = \langle t\cos t, t\sin t, t^2\rangle, \quad -\infty < t < \infty.$$

(a) (4 points) Find two different times t_1 and t_2 when the particle is at the point $P(0, \frac{\pi}{2}, \frac{\pi^2}{4})$.

LOOK AT Z-COMPONENT: $t^2 = \frac{\pi^2}{4} \longrightarrow \boxed{t = \pm \frac{\pi}{2}}$

THAT IS, $t_1 = -\frac{\pi}{2}$ & $t_2 = \frac{\pi}{2}$

CHECK: $\vec{r}(-\frac{\pi}{2}) = \langle 0, \frac{\pi}{2}, \frac{\pi^2}{4} \rangle$

$\vec{r}(\frac{\pi}{2}) = \langle 0, \frac{\pi}{2}, \frac{\pi^2}{4} \rangle$ ✓

(b) (4 points) Show that the particle is traveling in different directions each time it passes through the point P.

$\vec{r}\,'(t) = \langle \cos t - t\sin t, \sin t + t\cos t, 2t \rangle$

$$\boxed{\begin{aligned} \vec{r}\,'(-\tfrac{\pi}{2}) &= \langle -\tfrac{\pi}{2}, -1, -\pi \rangle \\ \vec{r}\,'(\tfrac{\pi}{2}) &= \langle -\tfrac{\pi}{2}, 1, \pi \rangle \end{aligned}}$$

$\vec{r}\,'(-\frac{\pi}{2}) \neq \vec{r}\,'(\frac{\pi}{2})$

DIFFERENT DIRECTIONS

2. Consider the two planes P_1 and P_2 given by the following equations.

$$x + 3z = 2y + 4 \qquad (P_1)$$

$$3y = 2z + 10 \qquad (P_1)$$

(a) (4 points) Are the planes P_1 and P_2 parallel, perpendicular, or neither? Show the calculation that supports your answer.

$P_1: \quad x - 2y + 3z = 4$

$P_2: \quad 3y - 2z = 10$

THE NORMAL VECTORS ARE NOT $\perp$

$$\boxed{\langle 1, -2, 3\rangle \cdot \langle 0, 3, -2\rangle = -12 \neq 0}$$

(b) (4 points) Give parametric equations for the line through the point $(1, 2, -1)$ that is parallel to both planes P_1 and P_2.

$\parallel$ TO PLANES $\Longleftrightarrow$ $\perp$ TO NORMAL VECTORS

$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 0 & 3 & -2 \end{vmatrix} = \langle -5, 2, 3\rangle$$

LINE : $\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle 1, 2, -1\rangle + t\langle -5, 2, 3\rangle$

$= \langle 1-5t, 2+2t, -1+3t\rangle$

$$\boxed{x = 1-5t, \quad y = 2+2t, \quad z = -1+3t}$$

(c) (4 points) Find the point at which the line you found in part (b) intersects the xz-plane.

XZ-PLANE $\longrightarrow$ $y = 0$ $\longrightarrow$ $2 + 2t = 0$

$t = -1$

$\vec{r}(-1) = \langle 6, 0, -4\rangle$ i.e. POINT $\boxed{(6, 0, -4)}$

3. (4 points) Show that the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)^2}{(x^2+y^2)^2}$$

SET $y=0$. $\lim\limits_{x\to 0} \dfrac{(x^2-0^2)^2}{(x^2+0^2)^2} = \lim\limits_{x\to 0} \dfrac{x^4}{x^4} = 1$

SET $y=x$. $\lim\limits_{x\to 0} \dfrac{(x^2-x^2)^2}{(x^2+x^2)^2} = \lim\limits_{x\to 0} \dfrac{0}{(2x^2)^2} = 0$

SINCE APPROACHING $(0,0)$ ALONG DIFFERENT LINES GIVES DIFFERENT LIMITS, THE LIMIT D.N.E.

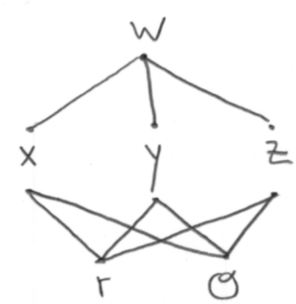
4. (4 points) Suppose

$$w = xy + yz + zx$$

and

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = r\theta$$

Find $\dfrac{\partial w}{\partial \theta}$ when $r = 2$ and $\theta = \dfrac{\pi}{2}$.



$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta}$$

$$= (y+z)(-r\sin\theta) + (x+z)(r\cos\theta) + (y+x)(r)$$

NOTE THAT WHEN $r=2$ AND $\theta = \frac{\pi}{2}$, WE HAVE $x=0$, $y=2$, $z=\pi$

$$\left.\frac{\partial w}{\partial \theta}\right|_{\substack{r=2\\ \theta=\frac{\pi}{2}}} = (2+\pi)(-2) + (0+\pi)(0) + (2+0)(2)$$

$$= -4 - 2\pi + 4 = \boxed{-2\pi}$$

5. Let $f(x,y,z) = x^2y^3 + e^{x+z} - 2y\sin z$.

(a) (4 points) Find the directional derivative of f at $P(0,1,0)$ in the direction toward the point $Q(3,5,-12)$, i.e. in the direction of the line segment from P to Q.

$$\vec{PQ} = \langle 3-0, 5-1, -12-0 \rangle = \langle 3, 4, -12 \rangle$$

$$|\vec{PQ}| = \sqrt{3^2 + 4^2 + 12^2} = \sqrt{169} = 13$$

$$\therefore \vec{u} = \frac{\vec{PQ}}{|\vec{PQ}|} = \left\langle \frac{3}{13}, \frac{4}{13}, \frac{-12}{13} \right\rangle$$ ← UNIT VECTOR!

$$D_{\vec{u}} f(0,1,0) = \nabla f(0,1,0) \cdot \vec{u}$$

NOTE: $\nabla f = \langle 2xy^3 + e^{x+z}, 3x^2y^2 - 2\sin z, e^{x+z} - 2y\cos z \rangle$

& $\nabla f(0,1,0) = \langle 1, 0, -1 \rangle$

$$\therefore D_{\vec{u}} f(0,1,0) = \langle 1, 0, -1 \rangle \cdot \left\langle \frac{3}{13}, \frac{4}{13}, \frac{-12}{13} \right\rangle = \frac{3}{13} + \frac{12}{13} = \boxed{\frac{15}{13}}$$

(b) (4 points) Find an equation of the tangent plane to the level surface $f(x,y,z) = 1$ at the point $P(0,1,0)$.

NOTE THAT $\nabla f(0,1,0)$ POINTS $\perp$ TO LEVEL SURFACE

i.e. $\langle 1, 0, -1 \rangle$ IS NORMAL TO TANGENT PLANE AT $(0,1,0)$

$$\langle 1, 0, -1 \rangle \cdot \langle x-0, y-1, z-0 \rangle = 0$$

$$\boxed{x - z = 0}$$

6. (4 points) Find the linear approximation to the function $f(x,y) = \tan^{-1}(xy^2)$ at the point $(1, 1, \frac{\pi}{4})$ and use it to approximate $f(0.9, 1.1)$.

$$f(x,y) \approx L(x,y) = f_x(1,1)(x-1) + f_y(1,1)(y-1) + f(1,1)$$

$$f_x = \frac{y^2}{1+(xy^2)^2} \qquad f_y = \frac{2xy}{1+(xy^2)^2}$$

$$f_x(1,1) = \frac{1}{2} \qquad f_y(1,1) = 1$$

$$\therefore \; L(x,y) = \frac{1}{2}(x-1) + y - 1 + \frac{\pi}{4}$$

$$L(0.9, 1.1) = \frac{1}{2}(-0.1) + 0.1 + \frac{\pi}{4} = \boxed{\frac{\pi}{4} + 0.05}$$

7. (4 points) Find $\dfrac{\partial z}{\partial y}$ when z is defined to be a function of x and y implicitly by the equation

$$yz + x \ln y = z^2.$$

$$F(x,y,z) = yz + x\ln y - z^2 = 0$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \boxed{-\frac{z + \frac{x}{y}}{y - 2z}}$$

LONG WAY: $\dfrac{\partial}{\partial y}\left[yz + x\ln y - z^2\right] = \dfrac{\partial}{\partial y}[0] = 0$

$$z + y\frac{\partial z}{\partial y} + \frac{x}{y} - 2z\frac{\partial z}{\partial y} = 0$$

$$(y - 2z)\frac{\partial z}{\partial y} = -z - \frac{x}{y}$$