NAME (printed) *ANSWER KEY*

NAME (signed)

MATH 203 Final Exam

December 22, 2014

Circle your section (for example, XX, Instructor, Days, Hours):

BB, Shell, M, W 9–10:40
DD, Shell, W 11–12:40
EE, Islam, M, W, 2–3:40
LL, Adamski, T, Th 9–10:40
LM, Jitsukawa, T, Th, 10–11:40
MM, Kapsack, T, F, 11–12:40
PP, Musser, T, Th, 2–3:40
RS, Bam, T, Th 4–5:40

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
Total	

Instructions: Complete every question in Part I (Questions 1–7), and answer THREE (3) COMPLETE questions from Part II (Questions 8–12). In the table above mark an **X** through the two questions that you omit. **Show all work.**

No calculators or other electric devices may be used. Answers are to be left in terms of $\sqrt{7}$, π, $\ln 3$, etc. when these can not be simplified. You have **2 hours and 15 minutes to complete the exam.**

PART I: Answer ALL questions in this part (10 points each)
Show all work and simplify all answers. **NO ELECTRONIC DEVICES!!!**

1. (a) Find an equation for the plane that contains the x-axis and the point $(5, 2, 3)$.
(b) Find parametric equations for the line through $(1, 1, 1)$ and $(1, 2, 3)$.
(c) Are the plane found in (a) and the line found in (b) parallel? Show work.

(a) $\vec{n} = \langle 1, 0, 0 \rangle \times \langle 5, 2, 3 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 5 & 2 & 3 \end{vmatrix} = \langle 0, -3, 2 \rangle$

$\boxed{-3y + 2z = 0}$

(b) $\vec{r}(t) = \langle 1, 1, 1 \rangle + t \langle 0, 1, 2 \rangle$

$\boxed{\begin{aligned} x(t) &= 1 \\ y(t) &= 1 + t \\ z(t) &= 1 + 2t \end{aligned}}$

(c) $\parallel$ IF $\vec{n} \cdot \vec{v} = 0$

$\downarrow$

$\langle 0, -3, 2 \rangle \cdot \langle 0, 1, 2 \rangle = -3 + 4 = 1 \neq 0$

$\boxed{\text{NO.}}$

2. The concentration of polutant in a thin film of poluted water at point (x, y) is $p(x, y) = xe^{x+2y}$.

(a) At the point $(2, -1)$, in which direction is the change per unit travelled of the concentration $p(x, y)$ decreasing the most rapidly?

(b) An ant crawling through the film is at position $(x, y) = (t^2 + t, t^3 - 2t)$ at time t. At what rate is the concentration at the ant's position changing per unit time at time $t = 1$?

(c) What is the rate of change in the concentration per unit distance travelled at the point $(4, -2)$ travelling towards the point $(3, 3)$?

(a) $\nabla p = \langle e^{x+2y} + xe^{x+2y}, \ 2xe^{x+2y} \rangle$

$\nabla p(2,-1) = \langle e^{2-2} + 2e^{2-2}, \ 4e^{2-2} \rangle = \langle 3, 4 \rangle$

$\boxed{\langle -3, -4 \rangle}$

(b) $\dfrac{dp}{dt} = \dfrac{\partial p}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial p}{\partial y} \cdot \dfrac{dy}{dt}$

$= (e^{x+2y} + xe^{x+2y})(2t+1) + (2xe^{x+2y})(3t^2 - 2)$

$\left.\dfrac{dp}{dt}\right|_{\substack{t=1 \\ x=2 \\ y=-1}} = (1+2)(2+1) + (4)(3-2) = 9 + 4 = \boxed{13}$

(c) $\vec{u} = \dfrac{1}{\sqrt{26}} \langle -1, 5 \rangle$. $\nabla p(4,-2) = \langle 5, 8 \rangle$

$D_{\vec{u}}\, p(5,8) = \dfrac{1}{\sqrt{26}} \langle -1, 5 \rangle \cdot \langle 5, 8 \rangle = \boxed{\dfrac{35}{\sqrt{26}}}$

3. Find all local maxima, local minima and saddle points of the graph of

$$f(x,y) = 2x^4 - x^2 + 3y^2 + 6y.$$

$f_x = 8x^3 - 2x = 0$

$2x(4x^2 - 1) = 0$

$2x(2x+1)(2x-1) = 0$

$x = 0, \pm\frac{1}{2}$

$f_y = 6y + 6 = 0$

$y = -1$

$\boxed{(-\tfrac{1}{2}, -1), (0,-1), (\tfrac{1}{2}, -1)}$

$f_{xx} = 24x^2 - 2$ $\quad f_{yy} = 6$ $\quad f_{xy} = 0$

$D(-\frac{1}{2}, -1) = (4)(6) - 0 > 0, \quad f_{xx}(-\frac{1}{2}) > 0$

$D(\frac{1}{2}, -1) = (4)(6) - 0 > 0, \quad \ldots$

$D(0, -1) = (-2)(6) - 0 < 0$

MIN $(-\frac{1}{2}, -1)$

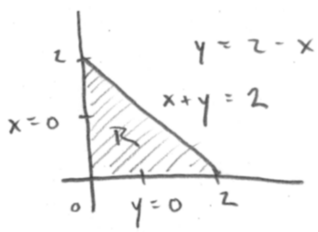
MIN $(\frac{1}{2}, -1)$

SADDLE $(0, -1)$

4. (a) Find the volume of the region in the first octant bounded by $x = 0$, $y = 0$, $x + y = 2$, $z = x + 3$, $z = y + 5$.

(b) Find an equation of the tangent plane to the graph of $z = \ln(e^x + e^{2y-4})$ at the point on the graph for which $(x, y) = (0, 2)$.

(a)



$$\int_0^2 \int_0^{2-x} \int_{x+3}^{y+5} dz\, dy\, dx$$

$$\int_0^2 \int_0^{2-x} y - x + 2 \; dy\, dx$$

$$= \int_0^2 \underbrace{\tfrac{1}{2}(2-x)^2 + (2-x)^2}_{\frac{3}{2}(2-x)^2} dx = -\tfrac{1}{2}(2-x)^3 \Big|_0^2 = \boxed{4}$$

$u = 2 - x$

$du = -dx$

(b) POINT : $(0, 2, \ln(2))$

$$F(x,y,z) = z - \ln(e^x + e^{2y-4}) = 0$$

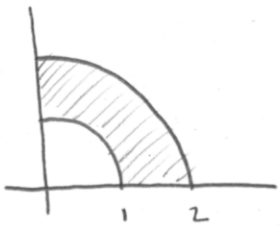
$$\nabla F = \left\langle \frac{-e^x}{e^x + e^{2y-4}}, \frac{-2e^{2y-4}}{e^x + e^{2y-4}}, 1 \right\rangle$$

$$\nabla F(0, 2, \ln(2)) = \langle -\tfrac{1}{2}, -1, 1 \rangle = \vec{n}$$

$$\langle -\tfrac{1}{2}, -1, 1 \rangle \cdot \langle x, y - 2, z - \ln(2) \rangle = 0$$

$$\boxed{-\tfrac{1}{2}x - y + z = \ln(2) - 2}$$

5. Find the mass of a lamina with density $\delta(x, y) = \dfrac{1 + \sqrt{x^2 + y^2}}{x^2 + y^2}$ which occupies the the portion of the annular ring $\{(x, y) : 1 \le x^2 + y^2 \le 4\}$ in the first quadrant.



$$\int_0^{\pi/2} \int_1^2 \frac{1 + r}{r^2} \cdot r \, dr \, d\theta$$

$$= \frac{\pi}{2} \int_1^2 \frac{1}{r} + 1 \, dr = \frac{\pi}{2} \Big[\ln r + r \Big]_1^2 = \boxed{\frac{\pi}{2} \left(\ln(2) + 1 \right)}$$

6. State, for each series, whether it converges absolutely, converges conditionally or diverges. Name a test which supports each conclusion and show the work to apply the test.

(a) $\sum_{n=1}^{\infty} \dfrac{(-1)^n(2^n)}{n+2^n}$ (b) $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n+2^n}$ (c) $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n+2}$

(a) $\boxed{\text{DIVERGES}}$: NOTE THAT $\lim_{n\to\infty} \left| \dfrac{(-1)^n 2^n}{n+2^n} \right| = \lim_{x\to\infty} \dfrac{2^x}{x+2^x} = 1$

BY L'HOPITAL'S RULE

(b) $\boxed{\text{CONVERGES ABSOLUTELY}}$:

$\sum \dfrac{1}{n+2^n} \leq \sum \left(\dfrac{1}{2}\right)^n = 1$ (BY COMPARISON TEST)

(c) $\boxed{\text{CONVERGES CONDITIONALLY}}$:

ALT. SERIES TEST: $\lim_{n\to\infty} \dfrac{1}{n+2} = 0$ ✓

BUT $\sum \dfrac{1}{n+2} = \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} + \dfrac{1}{6} + \cdots = \infty$

"HARMONIC SERIES"

7. Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(x+1)^n}{3^n \ln(n+2)}$. (Remember to check the endpoints, if applicable.)

IF $-3 < X + 1 < 3$ THEN SERIES CONVERGES

(COMPARE TO GEOMETRIC SERIES $\sum \left(\frac{X+1}{3}\right)^n$)

IF $X + 1 = 3$ THEN SERIES $= \sum \frac{1}{\ln(n+2)}$ DIVERGES

(COMPARE TO $\sum \frac{1}{n}$)

IF $X + 1 = -3$ THEN SERIES CONVERGES BY ALT. SERIES TEST.

$\therefore \quad -3 \leq X + 1 < 3$

$-4 \leq X < 2$ i.e. $\boxed{[-4, 2)}$

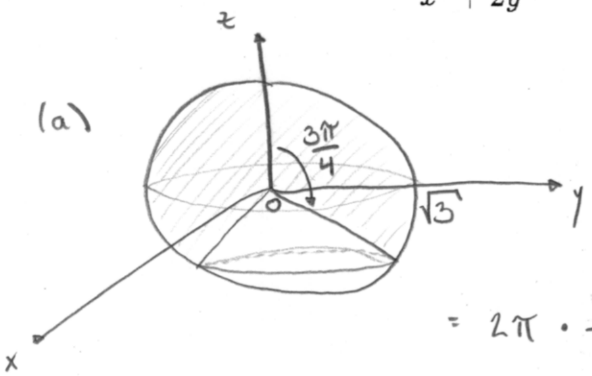
PART II: Answer any THREE COMPLETE questions (10 points each).

8. (a) Find the volume of the region inside the sphere $x^2 + y^2 + z^2 = 3$ and above the cone $z = -\sqrt{x^2 + y^2}$.

(b) Find the following limits or show they do not exist:

(i) $\lim_{(x,y)\to(0,0)} \frac{\sqrt{xy}}{x^2 + 2y^2}$ (ii) $\lim_{(x,y)\to(0,0)} \frac{\sqrt{xy}}{4 + x^2 + 2y^2}$

(a)



$$\int_0^{2\pi}\int_0^{3\pi/4}\int_0^{\sqrt{3}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \cdot \frac{\sqrt{3}^3}{3} \int_0^{3\pi/4} \sin\phi \, d\phi = 2\pi\sqrt{3}\left[-\cos\phi\right]_0^{3\pi/4}$$

$$= \boxed{2\pi\sqrt{3}\left(1 + \frac{\sqrt{2}}{2}\right)}.$$

(b) (i) $\boxed{\text{DNE}}$:

ALONG $y = 0$: $\lim_{x\to 0} \frac{0}{x^2 + 0} = 0$

ALONG $y = x$: $\lim_{x\to 0} \frac{x}{x^2 + 2x^2} = \lim_{x\to 0} \frac{1}{3x} = \infty$

(ii) $\boxed{0}$: THE FUNCTION IS CONTINUOUS AT $(0,0)$

$$\frac{\sqrt{(0)(0)}}{4 + (0)^2 + 2(0)^2} = 0 \quad \text{(NO PROBLEM!)}$$

9. (a) Use differentials (linear approximation) to approximate the volume of a (2.98 cm)$\times$(3.01 cm)$\times$(4.02 cm) rectangular box.

(b) Find an equation of the tangent plane to the graph of $\cos(x+2y+3z) = x+2y-e^{4z}$ at the point $(2,-1,0)$.

$\nabla f(3,3,4) = \langle 12, 12, 9 \rangle$

(a) let $f(x,y,z) = xyz$. $\nabla f = \langle yz, xz, xy \rangle$

$f(2.98, 3.01, 4.02) \approx f(3,3,4) + \nabla f(3,3,4) \cdot \langle 2.98-3, 3.01-3, 4.02-4 \rangle$

$\approx 36 + 12(-0.02) + 12(0.01) + 9(0.02)$

$\approx 36 - .24 + .12 + .18$

$\approx \boxed{36.06}$

(b) $F(x,y,z) = x + 2y - e^{4z} - \cos(x+2y+3z) = 0$

$\nabla F = \langle 1 + \sin(x+2y+3z), 2 + 2\sin(x+2y+3z), -4e^{4z} + 3\sin(x+2y+3z) \rangle$

$\nabla F(2,-1,0) = \langle 1, 2, -4 \rangle = \vec{n}$

$\langle 1, 2, -4 \rangle \cdot \langle x-2, y+1, z \rangle = 0$

$\boxed{x + 2y - 4z = 0}$

10. (a) Find the area of the portion of the surface $z = xy + 1$ which is inside the cylinder $x^2 + y^2 = 3$

(b) Graph, labelling the coordinates of any vertices: $x^2 - y^2 - 4y + z^2 = 4$.

(a) $$SA = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA \quad ; \quad f(x,y) = xy + 1$$

$$= \iint_R \sqrt{1 + x^2 + y^2}\, dA = \int_0^{2\pi}\int_0^{\sqrt{3}} \sqrt{1+r^2}\, r\, dr\, d\theta$$

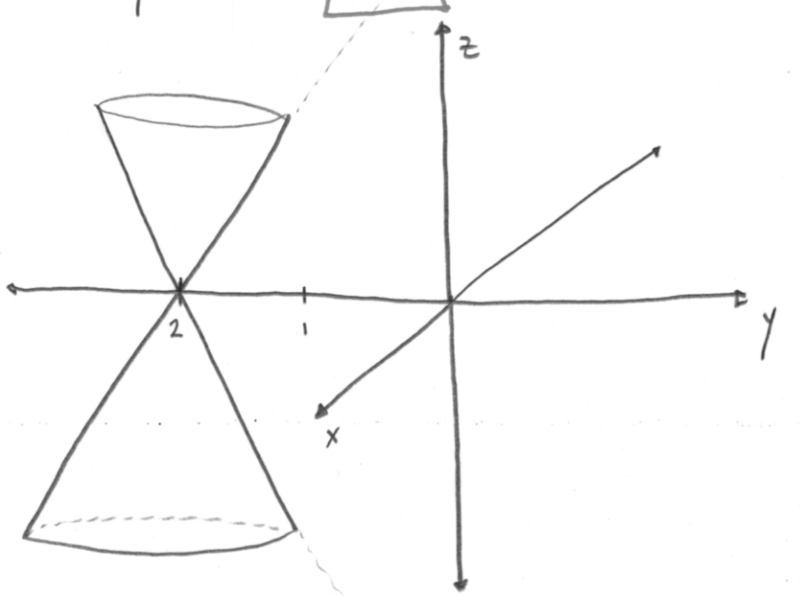
$u = 1 + r^2$

$du = 2r\, dr$

$$\pi \int_1^4 \sqrt{u}\, dr = \frac{2\pi}{3} u^{3/2} \Big|_1^4 = \boxed{\frac{14\pi}{3}}$$

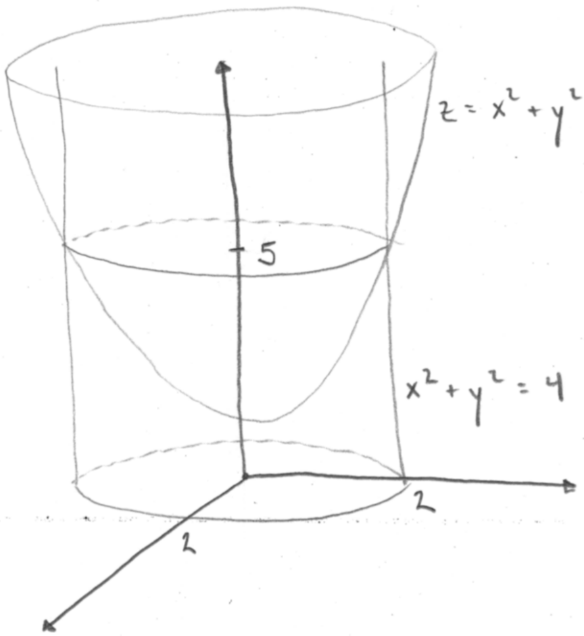
(b) $x^2 + z^2 = y^2 + 4y + 4$

$x^2 + z^2 = (y+2)^2$ $\boxed{\text{CONE}}$



11. Find the mass of the solid bounded by the surfaces $x^2 + y^2 = 4$, $z = x^2 + y^2 + 1$ and $z = 0$ and having density $\delta(x, y, z) = \sqrt{z}$.

$z = 4 + 1 = 5$



$$m = \iiint_S \rho(x,y,z)\, dV$$

$$= \int_0^{2\pi}\int_0^{2}\int_0^{r^2+1} \rho(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

CYLINDRICAL COORD.

$$= 2\pi \int_0^{2}\int_0^{r^2+1} \sqrt{z}\, r\, dz\, dr = \frac{4\pi}{3}\int_0^{2} (r^2+1)^{3/2}\, r\, dr$$

$u = r^2 + 1$

$\frac{1}{2}\, du = r\, dr$

$$= \frac{2\pi}{3}\int_1^{5} u^{3/2}\, du = \frac{4\pi}{15}\, u^{5/2}\Big|_1^{5}$$

$$= \boxed{\frac{4\pi}{15}\left(5^{5/2} - 1\right)}$$

12. (a) Find the first four nonzero terms of the Maclaurin series (i.e., the power series centered at zero) representation of the function $f(x) = e^{-x^2/2}$.

(b) Find the value $\int_0^1 f(x)\,dx$, accurate to the nearest hundredth. Justify that your answer has the required accuracy.

$f(0) = 1$

$f''(0) = -1$

$f^{(4)}(0) = 3$

$f^{(6)}(0) = -15$

$f'(x) = -x e^{-x^2/2}$

$f''(x) = -e^{-x^2/2} + x^2 e^{-x^2/2}$

$f'''(x) = x e^{-x^2/2} + 2x e^{-x^2/2} - x^3 e^{-x^2/2}$

$= 3x e^{-x^2/2} - x^3 e^{-x^2/2}$

$f^{(4)}(x) = 3e^{-x^2/2} - 3x^2 e^{-x^2/2} - 3x^2 e^{-x^2/2} + x^4 e^{-x^2/2}$

$= 3e^{-x^2/2} - 6x^2 e^{-x^2/2} + x^4 e^{-x^2/2}$

$f^{(5)}(x) = -3x e^{-x^2/2} - 12x e^{-x^2/2} + 6x^3 e^{-x^2/2} + 4x^3 e^{-x^2/2} - x^5 e^{-x^2/2}$

$= -15x e^{-x^2/2} + 10x^3 e^{-x^2/2} - x^5 e^{-x^2/2}$

$f^{(6)}(x) = -15e^{-x^2/2} + \ldots$

$$\boxed{f(x) \approx 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6}$$