Name: * ANSWER KEY * 6/19/2018

Math 202 Calculus II Quiz 2

Please show all work and box your final answers. If you need more room, you may use the backs of the pages. Calculators are not allowed. Good luck!

16 1. A small (12 oz.) Starbucks coffee contains 235 mg of caffeine. The half-life of caffeine in an adult human is approximately 5.5 hours. (The kidneys are responsible for filtering out the caffeine according to the *law of natural decay*.) How many hours after drinking one small Starbucks coffee does an adult human have only 40mg of caffeine remaining in their body?

NATURAL DECAY : $M(t) = M_0 e^{kt}$; $M_0 = M(0) = 235$

HALF-LIFE = 5.5 $\Rightarrow$ $M(5.5) = 235\, e^{k(5.5)} = \frac{1}{2}(235)$

$e^{5.5k} = \frac{1}{2} \rightarrow k = \frac{\ln(1/2)}{5.5}$

$\therefore M(t) = 235\, e^{\ln(1/2) \cdot \frac{t}{5.5}} = 235 \left(\frac{1}{2}\right)^{t/5.5}$

SOLVE FOR t: $M(t) = 235 \left(\frac{1}{2}\right)^{t/5.5} = 40$

$\left(\frac{1}{2}\right)^{t/5.5} = \frac{40}{235} = \frac{8}{47}$

$\frac{t}{5.5} \ln\left(\frac{1}{2}\right) = \ln\left(\frac{8}{47}\right) \longrightarrow t = \boxed{\frac{5.5 \ln\left(\frac{8}{47}\right)}{\ln(1/2)}} \approx 14$ HRS !!

2. (a) Given $f(x) = \sin^{-1}(x)$, state the domain and range of f. Also state $f'(x)$.

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$\text{Dom}(f) = [-1, 1]$, $\text{Ran}(f) = [-\frac{\pi}{2}, \frac{\pi}{2}]$, $f'(x) = \frac{1}{\sqrt{1-x^2}}$

6 (b) Given $g(x) = \cos^{-1}(x)$, state the domain and range of g. Also state $g'(x)$.

$\text{Dom}(g) = [-1, 1]$, $\text{Ran}(g) = [0, \pi]$, $g'(x) = \frac{-1}{\sqrt{1-x^2}}$

6 (c) Given $h(x) = \tan^{-1}(x)$, state the domain and range of h. Also state $h'(x)$.

$\text{Dom}(h) = (-\infty, \infty)$, $\text{Ran}(h) = (-\frac{\pi}{2}, \frac{\pi}{2})$, $h'(x) = \frac{1}{1+x^2}$

3. Evaluate the following expressions.

4 (a) $\sin^{-1}\left(\sin \frac{2\pi}{3}\right)$ $= \sin^{-1}\left(\sin \frac{\pi}{3}\right) = \boxed{\frac{\pi}{3}}$

$\frac{\pi}{3}$ $\underline{\underline{\text{IS}}}$ IN THE RANGE OF $\sin^{-1}$

4 (b) $\cos^{-1}\left(\cos \frac{-\pi}{6}\right)$ $= \cos^{-1}\left(\cos \frac{\pi}{6}\right) = \boxed{\frac{\pi}{6}}$

$\frac{\pi}{6}$ $\underline{\underline{\text{IS}}}$ IN THE RANGE OF $\cos^{-1}$

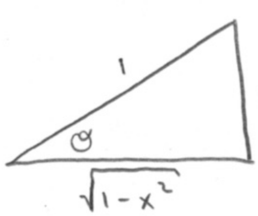
4 (c) $\tan^{-1}(\tan 0)$ $= \boxed{0}$

0 $\underline{\underline{\text{IS}}}$ IN THE RANGE OF $\tan^{-1}$

16 4. Simply the expression $\cot^2(\sin^{-1} x)$.

LET $\theta = \sin^{-1} x$. FIND $\cot^2 \theta$

$\sin \theta = x$



$\cot \theta = \frac{\text{ADJ}}{\text{OPP}} = \frac{\sqrt{1-x^2}}{x} \Rightarrow \cot^2 \theta = \boxed{\frac{1-x^2}{x^2}}$

16 5. Differentiate $y = \sin^{-1}(\sqrt{1-x})$. Simplify your answer.

$$y' = \frac{1}{\sqrt{1 - \sqrt{1-x}^{\,2}}} \cdot \frac{-1}{2\sqrt{1-x}}$$

$$y' = \frac{1}{\sqrt{1-(1-x)}} \cdot \frac{-1}{2\sqrt{1-x}} = \frac{1}{\sqrt{x}} \cdot \frac{-1}{2\sqrt{1-x}}$$

$$\boxed{y' = \frac{-1}{2\sqrt{x}\,\sqrt{1-x}}} \quad \text{or} \quad y' = \frac{-1}{2\sqrt{x-x^2}}$$

16 6. Prove that $\dfrac{d}{dx}\tanh^{-1} x = \dfrac{1}{1-x^2}$. *Hint: The following identities are true for all real t:*

$$\cosh^2 t - \sinh^2 t = 1, \qquad 1 - \tanh^2 t = \operatorname{sech}^2 t, \qquad \coth^2 - 1 = \operatorname{csch}^2 t\,.$$

LET $y = \text{TANH}^{-1} x$. FIND y'

$\text{TANH}\, y = x$ (DEF. OF INVERSE TANH)

$\text{SECH}^2 y \cdot y' = 1$ (IMPLICIT DIFFERENTIATION)

$$y' = \frac{1}{\text{SECH}^2 y} = \frac{1}{1 - \text{TANH}^2 y} = \frac{1}{1-x^2}$$

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