Name: *ANSWER KEY* 7/3/2018

Math 202 Calculus II Midterm Exam

Please show all work and box your final answers. If you need more room, you may use the backs of the pages. Calculators are not allowed. Good luck!

1. Let f be the one-to-one function $f(x) = 2x^3 + 3x^2 + 7x + 4$. Find $(f^{-1})'(4)$.

$f(0) = 4 \longleftrightarrow f^{-1}(4) = 0$

$f'(x) = 6x^2 + 6x + 7$

$f'(0) = 7$

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(0)} = \boxed{\frac{1}{7}}$$

INVERSE FUNCTION THM.

2. State the domain of the function $f(x) = \dfrac{e^{x-1}}{(\ln x)(1 - \ln x)}$.

SINCE THE FUNCTION CONTAINS $\ln x$, WE MUST HAVE $x > 0$ (DOMAIN OF $\ln x$) ①

ALSO, THE DENOMINATOR CANNOT EQUAL 0.

$\Rightarrow$ $\ln x \neq 0$ AND $1 - \ln x \neq 0$

$x \neq 1$ ②

$1 \neq \ln x$

$e \neq x$ ③

COMBINING ①, ②, & ③, WE HAVE $\boxed{(0,1) \cup (1,e) \cup (e,\infty)}$

3. For each of the following, find $\dfrac{dy}{dx}$.

(a) $y = x^{\sqrt{x}}(x^{\ln x})$

$$\ln y = \sqrt{x}\ln x + (\ln x)^2$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2\sqrt{x}}\ln x + \frac{\sqrt{x}}{x} + 2\ln x \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = \boxed{x^{\sqrt{x}}(x^{\ln x})\left[\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} + \frac{2\ln x}{x}\right]}$$

(b) $y = e^{\arctan^2 x}$

$$\frac{dy}{dx} = \boxed{e^{\text{ARCTAN}^2 x} \cdot 2\,\text{ARCTAN}\,x \cdot \frac{1}{1+x^2}}$$

(c) $y = \dfrac{5^x \sqrt[4]{x^3}}{x^7(x+1)^2}$

$$\ln y = x\ln 5 + \frac{3}{4}\ln x - 7\ln x - 2\ln(x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = \ln 5 + \frac{3}{4x} - \frac{7}{x} - \frac{2}{x+1}$$

$$\frac{dy}{dx} = \boxed{\frac{5^x \sqrt[4]{x^3}}{x^7(x+1)^2}\left[\ln 5 + \frac{3}{4x} - \frac{7}{x} - \frac{2}{x+1}\right]}$$

4. Suppose a colony of bacteria grows according to the law of *natural decay*. Initially, the colony contains 2400 bacteria, and 3 days later the colony contains 6000 bacteria. How long will it take the population to grow from 2400 bacteria to 9600 bacteria?

$$P(t) = P_0 e^{kt} = 2400 e^{kt}$$

$$P(3) = 2400 e^{3k} = 6000$$

$$e^{3k} = \frac{5}{2}$$

$$3k = \ln\left(\frac{5}{2}\right)$$

$$k = \frac{1}{3}\ln\left(\frac{5}{2}\right)$$

$$P(t) = 2400\left(\frac{5}{2}\right)^{t/3} = 9600$$

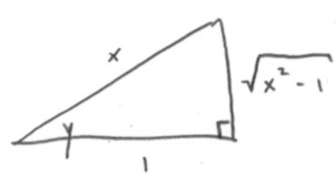
$$\left(\frac{5}{2}\right)^{t/3} = 4$$

$$\frac{t}{3}\ln\left(\frac{5}{2}\right) = \ln 4$$

$$t = \boxed{\frac{3\ln 4}{\ln\left(\frac{5}{2}\right)}}$$

5. Derive the derivative $\dfrac{d}{dx}\left[\sec^{-1} x\right]$.

LET $y = \sec^{-1} x$. FIND y'.



$$\sec y = x$$

$$\sec y \tan y \cdot y' = 1$$

$$y' = \frac{1}{\sec y \tan y} = \boxed{\frac{1}{x\sqrt{x^2 - 1}}}$$

✓

6. Evaluate the following limits.

(a) $\lim_{x\to\infty} \frac{\cosh(1/x)}{e^{1/x}} = \lim_{x\to\infty} \frac{e^{1/x} + e^{-1/x}}{2e^{1/x}} = \frac{1+1}{2\cdot 1}$

$= \boxed{1}$

(b) $\lim_{x\to 0} \frac{5^x - 4^x}{3^x - 2^x} \quad : \frac{0}{0}$

$\xrightarrow{\text{L'HO}} = \lim_{x\to 0} \frac{5^x \ln 5 - 4^x \ln 4}{3^x \ln 3 - 2^x \ln 2} = \boxed{\frac{\ln 5 - \ln 4}{\ln 3 - \ln 2}}$

or $\frac{\ln(5/4)}{\ln(3/2)}$

(c) $\lim_{x\to 0^+} (\cos x)^{1/x^2} \quad : 1^\infty$

$= \text{EXP}\left[\lim_{x\to 0^+} \frac{\ln(\cos x)}{x^2}\right] : \frac{0}{0}$

$\xrightarrow{\text{L'HO}} \text{EXP}\left[\lim_{x\to 0^+} \frac{\frac{-\sin x}{\cos x}}{2x}\right] = \text{EXP}\left[\lim_{x\to 0^+} \frac{-\tan x}{2x}\right] : \frac{0}{0}$

$\xrightarrow{\text{L'HO}} \text{EXP}\left[\lim_{x\to 0^+} \frac{-\sec^2 x}{2}\right] = \text{EXP}\left[-\tfrac{1}{2}\right] = \boxed{e^{-1/2}}$ or $\frac{1}{\sqrt{e}}$

7. Evaluate the following integrals.

(a) $\int x^2 e^x \, dx$ LET $u = x^2$ $v = e^x$

$du = 2x\,dx$ $dv = e^x\,dx$

$= x^2 e^x - 2\int x e^x \, dx$ Let $u = x$ $v = e^x$

$du = dx$ $dv = e^x\,dx$

$= x^2 e^x - 2\left(x e^x - \int e^x \, dx\right)$

$= \boxed{x^2 e^x - 2x e^x + 2e^x + C}$

OR $e^x(x^2 - 2x + 2) + C$

(b) $\int \tan^5 \pi\theta \, d\theta$ LET $u = \pi\theta$

$du = \pi\, d\theta \longrightarrow \frac{1}{\pi} du = d\theta$

$\rightsquigarrow \frac{1}{\pi}\int \tan^5 u \, du = \frac{1}{\pi}\int \tan^4 u \tan u \, du$

$= \frac{1}{\pi}\int (\sec^2 u - 1)^2 \tan u \, du = \frac{1}{\pi}\int (\sec^4 u - 2\sec^2 u + 1)\tan u \, du$

$= \frac{1}{\pi}\underbrace{\int \sec^4 u \tan u \, du}_{\substack{w = \sec u \\ dw = \sec u \tan u \, du}} - \frac{2}{\pi}\underbrace{\int \sec^2 u \tan u \, du}_{\substack{w = \tan u \\ dw = \sec^2 u \, du}} + \frac{1}{\pi}\underbrace{\int \tan u \, du}_{\substack{w = \cos u \\ -dw = \sin u \, du}}$

$= \frac{1}{4\pi}\sec^4 u - \frac{1}{\pi}\tan^2 u + \frac{1}{\pi}\ln|\sec u| + C$

$\rightsquigarrow \boxed{\frac{1}{4\pi}\sec^4 \pi\theta - \frac{1}{\pi}\tan^2 \pi\theta + \frac{1}{\pi}\ln|\sec \pi\theta| + C}$

8. Evaluate the following integrals.

(a) $\displaystyle\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx$

LET $x = 4\sin\theta$

$dx = 4\cos\theta\, d\theta$

BOUNDS: WHEN $x = 2\sqrt{3}$,

$\sin\theta = \frac{x}{4} = \frac{\sqrt{3}}{2}$

$\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

WHEN $x = 0$

$\sin\theta = \frac{x}{4} = 0$

$\theta = \sin^{-1}(0) = 0$

$$\rightsquigarrow \int_0^{\pi/3} \frac{(4\sin\theta)^3}{\sqrt{16-16\sin^2\theta}}\, 4\cos\theta\, d\theta$$

$$= 64\int_0^{\pi/3} \sin^3\theta\, d\theta = 64\int_0^{\pi/3} (1-\cos^2\theta)\sin\theta\, d\theta$$

LET $u = \cos\theta$

$-du = \sin\theta\, d\theta$

$$\rightsquigarrow -64\int_1^{1/2} (1-u^2)\,du = 64\int_{1/2}^{1} (1-u^2)\,du$$

$$= 64\left[u - \tfrac{1}{3}u^3\right]_{1/2}^{1} = 64\left[\left(1-\tfrac{1}{3}\right) - \left(\tfrac{1}{2} - \tfrac{1}{24}\right)\right] = 64\left(\frac{5}{24}\right) = \frac{40}{3}$$

(b) $\displaystyle\int \frac{x^2+2x-1}{x^3-x}\,dx$

① PROPER ✓

② $Q(x) = x(x^2-1) = x(x+1)(x-1)$

③ PFD : $\dfrac{x^2+2x-1}{x(x+1)(x-1)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$

$$x^2+2x-1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$$

$x=0$: $-1 = -A \rightarrow A = 1$

$x=1$: $2 = 2C \rightarrow C = 1$

$x=-1$: $-2 = 2B \rightarrow B = -1$

$$\therefore \int \frac{x^2+2x-1}{x^3-x}\,dx = \int \frac{1}{x} - \frac{1}{x+1} + \frac{1}{x-1}\,dx = \boxed{\ln|x| - \ln|x+1| + \ln|x-1| + C}$$

3.c. (ALTERNATIVE)

$$y = \frac{5^x x^{3/4}}{x^{28/4}(x+1)^2} = \frac{5^x}{x^{25/4}(x+1)^2}$$

$$\frac{dy}{dx} = \frac{x^{25/4}(x+1)^2 \frac{d}{dx}\left[5^x\right] - 5^x \frac{d}{dx}\left[x^{25/4}(x+1)^2\right]}{\left(x^{25/4}(x+1)^2\right)^2}$$

$$\frac{dy}{dx} = \boxed{\frac{x^{25/4}(x+1)^2\, 5^x \ln 5 - 5^x\left[\frac{25}{4}x^{21/4}(x+1)^2 + x^{25/4}\cdot 2(x+1)\right]}{\left(x^{25/4}(x+1)^2\right)^2}}$$

7.b (ALTERNATIVE)

$$\int \tan^5 \pi\theta\, d\theta \rightsquigarrow \frac{1}{\pi}\int (\sec^2 u - 1)^2 \tan u\, du$$

LET $w = \sec u$

$dw = \sec u \tan u\, du \implies \tan u\, du = \frac{1}{\sec u}\, dw$

$= \frac{1}{w}\, dw$

$$\rightsquigarrow \frac{1}{\pi}\int (w^2-1)\frac{1}{w}\, dw = \frac{1}{\pi}\int w^3 - 2w - \frac{1}{w}\, dw$$

$$= \frac{1}{\pi}\left[\frac{1}{4}w^4 - w^2 - \ln|w|\right] + C$$

$$\rightsquigarrow \boxed{\frac{1}{\pi}\left[\frac{1}{4}\sec^4(\pi\theta) - \sec^2(\pi\theta) - \ln|\sec(\pi\theta)|\right] + C}$$