MATH 202

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§6.6

[2] (a) NOT IMPROPER, $\frac{1}{2x-1}$ IS CONTINUOUS ON $[1,2]$

& ONLY DISCONTINUOUS AT $x = \frac{1}{2}$

(b) IMPROPER BECAUSE OF DISCONTINUITY AT $x = \frac{1}{2} \in [0,1]$

(c) IMPROPER BECAUSE OF INFINITE LIMITS OF INTEGRATION.

(d) IMPROPER BECAUSE OF INFINITE DISCONTINUITY AT THE LOWER LIMIT OF INTEGRATION, $x = 1$.

[5]

$$\lim_{t\to\infty} \int_1^t \frac{1}{(3x+1)^2}\,dx \qquad \text{LET } u = 3x+1, \quad du = 3\,dx$$

$$\longrightarrow \frac{1}{3}\int \frac{1}{u^2}\,du = \frac{-1}{3u} \longrightarrow \left.\frac{-1}{3(3x+1)}\right|_1^t$$

$$\longrightarrow \lim_{t\to\infty} \frac{1}{12} - \frac{1}{9t+3} = \boxed{\frac{1}{12} \text{ (CONVERGENT)}}$$

[9]

$$\lim_{t\to\infty} \int_4^t e^{-y/2}\,dy = \lim_{t\to\infty} \left. -2e^{-y/2} \right|_4^t$$

$$= \lim_{t\to\infty} 2e^{-2} - 2e^{-t/2} = \boxed{2e^{-2} \text{ or } \frac{2}{e^2} \text{ (CONVERGENT)}}$$

[11] $$\lim_{t\to\infty}\int_{2\pi}^{t} \sin\theta \, d\theta = \lim_{t\to\infty} -\cos\theta \Big|_{2\pi}^{t} = \lim_{t\to\infty} 1-\cos(t)$$

DOES NOT CONVERGE $\Rightarrow$ $\boxed{\text{DIVERGES}}$

[21] $$\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\,dx = \int_{-\infty}^{0} \frac{x^2}{9+x^6}\,dx + \int_{0}^{\infty} \frac{x^2}{9+x^6}\,dx$$

AS LONG AS BOTH IMPROPER INTEGRALS ON THE RIGHT CONVERGE.

$$= \lim_{t\to-\infty}\int_{t}^{0} \frac{x^2}{9+x^6}\,dx + \lim_{s\to\infty}\int_{0}^{s} \frac{x^2}{9+x^6}\,dx$$

LET $u = x^3$

$du = 3x^2$

NOTE THAT $u \to \infty$ AS $t \to \infty$

AND $u \to -\infty$ AS $t \to -\infty$

AND

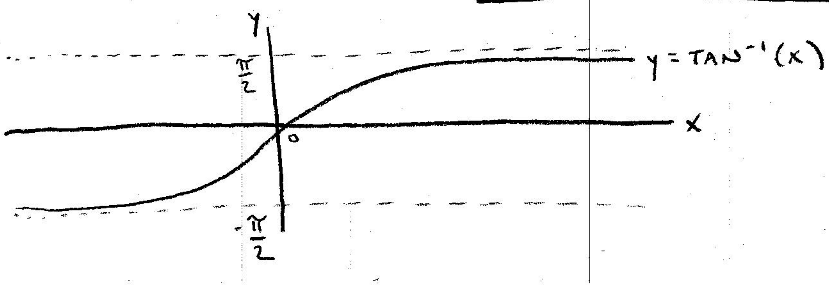
$x = 0 \Rightarrow u = 0$

$$= \lim_{t\to-\infty} \frac{1}{3}\int_{t}^{0} \frac{1}{9+u^2}\,du + \lim_{s\to\infty} \frac{1}{3}\int_{0}^{s} \frac{1}{9+u^2}\,du$$

$$= \lim_{t\to-\infty} \frac{1}{9}\tan^{-1}\left(\frac{u}{3}\right)\Big|_{t}^{0} + \lim_{s\to\infty} \frac{1}{9}\tan^{-1}\left(\frac{u}{3}\right)\Big|_{0}^{s}$$

$$= \lim_{t\to-\infty} -\frac{1}{9}\tan^{-1}\left(\frac{t}{3}\right) + \lim_{s\to\infty} \frac{1}{9}\tan^{-1}\left(\frac{s}{3}\right)$$

$$= -\frac{1}{9}\left(-\frac{\pi}{2}\right) + \frac{1}{9}\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi}{9} \quad (\text{CONVERGES})}$$



23 $\lim_{t\to 0^+} \int_t^1 \frac{3}{x^5}\,dx = \lim_{t\to 0^+} -\frac{3}{4x^4}\Big|_t^1 = \frac{3}{4}\lim_{t\to 0^+} \underbrace{\frac{1}{t^4}}_{\to\infty} - 1$ $\boxed{\text{DIVERGES}}$

25 $\lim_{t\to -2^+} \int_t^{14} \frac{dx}{\sqrt[4]{x+2}} = \lim_{t\to -2^+} \frac{4}{3}(x+2)^{3/4}\Big|_t^{14}$

$= \frac{4}{3}\lim_{t\to -2^+} 16^{3/4} - (t+2)^{3/4} = \frac{4}{3}(8-0) = \boxed{\frac{32}{3} \quad (\text{CONVERGES})}$

27 NOTE THAT THIS INTEGRAND HAS AN INFINITE DISCONTINUITY AT $x=1$

$\lim_{t\to 1^-} \int_0^t (x-1)^{-1/5}\,dx + \lim_{s\to 1^+} \int_s^{33} (x-1)^{-1/5}\,dx$

$= \lim_{t\to 1^-} \frac{5}{4}(x-1)^{4/5}\Big|_0^t + \lim_{s\to 1^+} \frac{5}{4}(x-1)^{4/5}\Big|_s^{33}$

$= \lim_{t\to 1^-} \frac{5}{4}\Big(\underbrace{(t-1)^{4/5}}_{\to 0} - 1\Big) + \lim_{s\to 1^+} \frac{5}{4}\Big(16 - \underbrace{(s-1)^{4/5}}_{\to 0}\Big)$

$= -\frac{5}{4} + \frac{80}{4} = \boxed{-\frac{75}{4} \quad (\text{CONVERGES})}$

29 NOTE: INTEGRAND HAS INFINITE DISCONTINUITY AT $x = 0$.

$$\lim_{t \to 0^-} \int_{-1}^{t} \frac{e^x}{e^x - 1}\,dx + \lim_{s \to 0^+} \int_{s}^{1} \frac{e^x}{e^x - 1}\,dx$$

$$= \lim_{t \to 0^-} \ln|e^x - 1| \Big|_{-1}^{t} + \lim_{s \to 0^+} \ln|e^x - 1| \Big|_{s}^{1}$$

$$= \lim_{t \to 0^-} \Big[\underbrace{\ln|e^t - 1|}_{\to \infty} - \ln|e^{-1} - 1| \Big] + \lim_{s \to 0^+} \Big[\ln(e-1) - \underbrace{\ln(e^s - 1)}_{\to -\infty} \Big]$$

$\therefore$ $\boxed{\text{DIVERGES}}$

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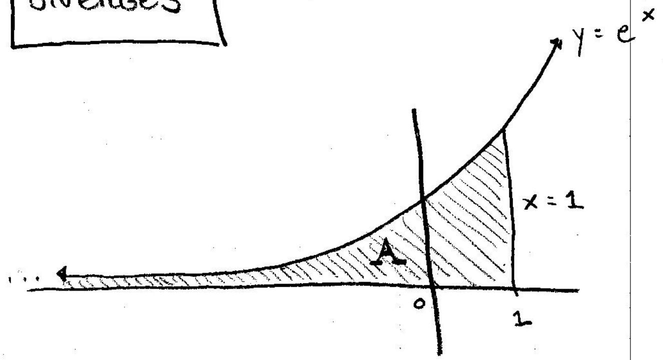

$$\text{AREA } A = \int_{-\infty}^{1} e^x\,dx = \lim_{t \to -\infty} \int_{t}^{1} e^x\,dx = \lim_{t \to -\infty} e^x \Big|_{t}^{1}$$

$$= \lim_{t \to -\infty} e - \underbrace{e^t}_{\to 0} = \boxed{e}$$

41 OBSERVE THAT $-1 \le \cos x \le 1$ FOR ALL x

$$\Rightarrow \quad 0 \le \cos^2 x \le 1 \quad \text{FOR ALL } x$$

$$\Rightarrow \quad 0 \le \frac{\cos^2 x}{1+x^2} \le \frac{1}{1+x^2} \quad \text{FOR ALL } x$$

(IN THE LAST STEP, I AM USING THE FACT THAT $1+x^2 > 0$ FOR ALL x.)

$$\text{NOW, } \int_1^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \tan^{-1}(x)\Big|_1^t$$

$$= \lim_{t\to\infty} \tan^{-1}(t) - \tan^{-1}(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$ FINITE (i.e. CONVERGES)

$\therefore$ BY THE COMPARISON TEST, $\int_1^\infty \frac{\cos^2 x}{1+x^2}\,dx$ $\boxed{\text{CONVERGES}}$

43 OBSERVE THAT $0 \le \frac{1}{x+e^{2x}} \le \frac{1}{e^{2x}} = e^{-2x}$ FOR $x \ge 1$.

$$\text{AND } \int_1^\infty e^{-2x}\,dx = \lim_{t\to\infty} \int_1^t e^{-2x}\,dx = \lim_{t\to\infty} -\frac{1}{2}e^{-2x}\Big|_1^t$$

$$= \lim_{t\to\infty} \frac{1}{2}e^{-2} - \underbrace{\frac{1}{2}e^{-2t}}_{\to 0} = \frac{1}{2}e^{-2} \text{ OR } \frac{1}{2e^2}$$ CONVERGES

$\therefore$ BY THE COMPARISON TEST, $\int_1^\infty \frac{1}{x+e^{2x}}\,dx$ $\boxed{\text{CONVERGES}}$

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$$\lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt{x}(1+x)}\,dx + \lim_{s\to\infty}\int_1^s \frac{1}{\sqrt{x}(1+x)}\,dx$$

LET $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$x = 1 \Rightarrow u = 1$

$x \to 0 \Rightarrow u \to 0$

$x \to \infty \Rightarrow u \to \infty$

$$= \lim_{t\to 0^+} 2\int_t^1 \frac{1}{1+u^2}\,du + \lim_{s\to\infty} 2\int_1^s \frac{1}{1+u^2}\,du$$

$$= 2\left[\lim_{t\to 0^+} \tan^{-1}(u)\Big|_t^1 + \lim_{s\to\infty} \tan^{-1}(u)\Big|_1^s\right]$$

$$= 2\left[\lim_{t\to 0^+}\left(\frac{\pi}{4} - \underbrace{\tan^{-1}(t)}_{\to 0}\right) + \lim_{s\to\infty}\left(\underbrace{\tan^{-1}(s)}_{\to \frac{\pi}{2}} - \frac{\pi}{4}\right)\right]$$

$$= 2\left[\frac{\pi}{4} + \frac{\pi}{2} - \frac{\pi}{4}\right] = \boxed{\pi}$$

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IF $p = 1$ THEN $\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} \ln(x)\Big|_t^1 = -\infty$

$\Rightarrow$ DIVERGES.

IF $p \neq 1$ THEN $\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+}\int_t^1 x^{-p}\,dx$

$$= \lim_{t\to 0^+} \frac{1}{1-p} x^{1-p}\Big|_t^1 = \frac{1}{1-p}\left[1 - \lim_{t\to 0^+} t^{1-p}\right]$$

NOW, IF $p < 1$ THEN $1-p > 0$ (POSITIVE)

$\Rightarrow \lim_{t \to 0^+} t^{1-p} = 0$ AND THE IMPROPER INTEGRAL CONVERGES

TO $\frac{1}{1-p}$.

IF $p > 1$, THEN $1-p < 0$ (NEGATIVE) [& THUS $p-1 > 0$ POSITIVE]

$\Rightarrow \lim_{t \to 0^+} t^{1-p} = \lim_{t \to 0^+} \frac{1}{t^{p-1}} = \infty$ AND THE IMPROPER INTEGRAL DIVERGES.

SUMMARY:

$\boxed{\text{IF } p < 1 \text{, THE INTEGRAL CONVERGES TO } \frac{1}{1-p}. \quad \text{IF } p \geq 1 \text{, THE INTEGRAL DIVERGES.}}$

[51] (a) $\int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{\infty} x\,dx$ PROVIDED BOTH THE RIGHT-SIDE INTEGRALS CONVERGE.

$= \lim_{t \to -\infty} \int_{t}^{0} x\,dx + \lim_{s \to \infty} \int_{0}^{s} x\,dx = \lim_{t \to -\infty} \frac{1}{2}x^2 \Big|_{t}^{0} + \lim_{s \to \infty} \frac{1}{2}x^2 \Big|_{0}^{s}$

$= \lim_{t \to -\infty} -\frac{1}{2}t^2 + \lim_{t \to \infty} \frac{1}{2}s^2 = -\infty + \infty$ DIVERGES ✓

(b) $\lim_{t \to \infty} \int_{-t}^{t} x\,dx = \lim_{t \to \infty} \frac{1}{2}x^2 \Big|_{-t}^{t} = \lim_{t \to \infty} \frac{1}{2}\left((t)^2 - (-t)^2\right)$

$= \lim_{t \to \infty} \frac{1}{2}(0) = 0$ ✓