MATH 202

§6.2

ODD!

[1] $\int \sin^3 x \cos^2 x \, dx = \int (\sin^2 x) \cos^2 x \sin x \, dx$

$= \int (1-\cos^2 x)\cos^2 x \sin x \, dx = \int (\cos^2 x - \cos^4 x) \sin x \, dx$

LET $u = \cos x$ $\longrightarrow -\int u^2 - u^4 \, du = \frac{1}{5}u^5 - \frac{1}{3}u^3$

$du = -\sin x \, dx$

$\longrightarrow \boxed{\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C}$

ODD!

[3] $\int_{\pi/2}^{3\pi/4} \sin^5 x \cos^3 x \, dx = \int_{\pi/2}^{3\pi/4} (1-\cos^2\theta)^2 \cos^3 x \sin x \, dx$

LET $u = \cos(x)$ $\quad x = \frac{\pi}{2} \Rightarrow u = 0$

$du = -\sin(x)\,dx$ $\quad x = \frac{3\pi}{4} \Rightarrow u = \frac{\sqrt{2}}{2}$

$\longrightarrow -\int_0^{\sqrt{2}/2} (1-u^2)^2 u^3 \, du = -\int_0^{\sqrt{2}/2} (u^3 - 2u^5 + u^7)\, du$

$= -\frac{1}{4}u^4 + \frac{1}{3}u^6 - \frac{1}{8}u^8 \Big|_0^{\sqrt{2}/2} = -\frac{1}{4}\left(\frac{1}{4}\right) + \frac{1}{3}\left(\frac{1}{8}\right) - \frac{1}{8}\left(\frac{1}{16}\right)$

$= -\frac{1}{16}\cdot\frac{24}{24} + \frac{1}{24}\cdot\frac{16}{16} - \frac{1}{128}\cdot\frac{3}{3} = \boxed{\frac{-11}{384}}$

$\boxed{5}$ $\displaystyle\int_0^{\pi/2} \cos^2\theta \, d\theta = \int_0^{\pi/2} \frac{1}{2}(1+\cos(2\theta))\, d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\pi/2}$

$\displaystyle = \frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \boxed{\frac{\pi}{4}}$

$\boxed{7}$ $\displaystyle\int_0^{\pi} \sin^4(3t)\, dt = \int_0^{\pi} (\sin^2(3t))^2\, dt = \int_0^{\pi} \left[\frac{1}{2}(1-\cos(6t)\right)^2 dt$

$\displaystyle = \frac{1}{4}\int_0^{\pi} 1 - 2\cos(6t) + \cos^2(6t)\; dt = \frac{1}{4}\left[\left[t - \frac{1}{3}\sin(6t)\right]_0^{\pi} + \int_0^{\pi} \cos^2(6t)\, dt\right]$

$\displaystyle = \frac{1}{4}\left[\pi + \int_0^{\pi} \frac{1}{2}(1+\cos(12t))\, dt\right] = \frac{1}{4}\left[\pi + \frac{1}{2}\left[t + \frac{1}{12}\sin(12t)\right]_0^{\pi}\right]$

$\displaystyle = \frac{1}{4}\left(\pi + \frac{1}{2}(\pi)\right) = \frac{\pi}{4} + \frac{\pi}{8} = \boxed{\frac{3\pi}{8}}$

$\boxed{11}$ $\displaystyle\int_0^{\pi/4} \sin^4 x \cos^2 x\, dx = \int_0^{\pi/4} \sin^2 x\, (\sin x \cos x)^2\, dx$

LET $u = \sin(2x)$

$du = 2\cos(2x)\, dx$

$\displaystyle = \int_0^{\pi/4} \frac{1}{2}(1-\cos(2x))\left(\frac{1}{2}\sin(2x)\right)^2 dx$

$\displaystyle = \frac{1}{8}\int_0^{\pi/4} \underline{\sin^2(2x)} - \sin^2(2x)\cos(2x)\, dx = \frac{1}{8}\int_0^{\pi/4} \underline{\frac{1}{2}(1-\cos(4x))}dx - \frac{1}{8}\int_0^{\pi/4} \sin^2(2x)\cos(2x)dx$

$\displaystyle = \frac{1}{16}\left[x - \frac{1}{4}\sin(4x)\right]_0^{\pi/4} - \frac{1}{48}\left[\sin^3(2x)\right]_0^{\pi/4}$

$\displaystyle = \frac{1}{16}\left(\frac{\pi}{4}\right) - \frac{1}{48}(1) = \frac{\pi}{64} - \frac{1}{48} = \boxed{\frac{3\pi - 4}{192}}$

15 $\displaystyle\int \frac{1 - \sin x}{\cos x}\,dx = \int \frac{1}{\cos x} - \frac{\sin x}{\cos x}\,dx$

$\displaystyle = \int \sec x\,dx - \int \tan x\,dx = \ln|\sec x + \tan x| - \ln|\sec x| + C$

$\displaystyle = \ln\left|\frac{\sec x + \tan x}{\sec x}\right| + C = \boxed{\ln|1 + \sin x| + C}$

19 $\displaystyle\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \boxed{\tan x - x + C}$

21 $\displaystyle\int \sec^6 t\,dt = \int (\sec^2 t)^2 \sec^2 t\,dt = \int (\tan^2 t + 1)^2 \sec^2 t\,dt$

Let $u = \tan t$

$du = \sec^2 t\,dt$

$\displaystyle\longrightarrow \int (u^2+1)^2\,du = \int u^4 + 2u^2 + 1\,du$

$\displaystyle = \frac{1}{5}u^5 + \frac{2}{3}u^3 + u \longrightarrow \boxed{\frac{1}{5}\tan^5 t + 2\tan^2 t + \tan t + C}$

23 $\displaystyle\int_0^{\pi/3} \tan^5 x \sec^4 x\,dx = \int_0^{\pi/3} \tan^5 x\,(\tan^2 x + 1)\sec^2 x\,dx$

Let $u = \tan x$

$du = \sec^2 x\,dx$

$\displaystyle\longrightarrow \int_0^{\sqrt{3}} u^5(u^2+1)\,du = \frac{1}{8}u^8 + \frac{1}{6}u^6 \Big|_0^{\sqrt{3}} = \frac{1}{8}(81) + \frac{1}{6}(27) =$

$\displaystyle = \frac{81}{8} + \frac{9}{2}\cdot\frac{4}{4} = \boxed{\frac{117}{8}}$

25 $\int \tan^3 x \sec x = \int (\tan^2 x) \tan x \sec x \, dx$

$= \int (\sec^2 x - 1) \tan x \sec x \, dx$

$$= \underbrace{\int \sec^2 x \cdot \tan x \sec x \, dx}_{(1)} - \underbrace{\int \tan x \sec x \, dx}_{(2)}$$

(1): Let $u = \sec x$ $\longrightarrow \int u^2 \, du = \frac{1}{3} u^3$

$du = \tan x \sec x$

$\longrightarrow \frac{1}{3} \sec^3 x$

(2): Let $u = \sec x$ $\quad v = \ln|\sec x|$

$du = \tan x \sec x \, dx$ $\quad dv = \tan x \, dx$

$\longrightarrow \sec x \ln|\sec x| - \int \ln|\sec x| \tan x \sec x \, dx$

Let $u = \sec x$

$du = \tan x \sec x \, dx$

$\longrightarrow \int \ln|u| \, du = u \ln|u| - u$

$$= \underbrace{\sec x \ln|\sec x| - \sec x \ln|\sec x|}_{\text{Z E R O ! ! ! !}} + \sec x$$

$\therefore \int \tan^3 x \sec x \, dx = (1) - (2) = \boxed{\frac{1}{3} \sec^3 x - \sec x + C}$

27 $\int \tan^5 x\, dx = \int \tan^3 x \,(\sec^2 x - 1)\, dx$

$$= \underbrace{\int \tan^3 x \sec^2 x \, dx}_{(1)} - \underbrace{\int \tan^3(x)\, dx}_{(2)}$$

(1) let $u = \tan x$ $\longrightarrow \int u^3 du = \frac{1}{4}u^4 \longrightarrow \frac{1}{4}\tan^4 x$

$du = \sec^2 x\, dx$

(2) $\int \tan^3 x\, dx = \int \tan x\,(\sec^2 x - 1)\, dx$

$$= \int \tan x \sec^2 x\, dx - \int \tan(x)\, dx$$

$$= \frac{1}{2}\tan^2(x) - \ln|\sec x|$$

$\therefore \int \tan^5 x\, dx = (1) - (2) = \boxed{\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + C}$

32 $\int \csc^4 x \cot^6 x\, dx = \int (\csc^2 x) \cot^6 x \csc^2 x\, dx$

$$= \int (1 + \cot^2 x)\cot^6 x \csc^2 x\, dx$$

let $u = \cot x$

$du = -\csc^2 x\, dx$

$\longrightarrow -\int (1+u^2)u^6 du = -\frac{1}{7}u^7 - \frac{1}{9}u^9$

$\longrightarrow \boxed{-\frac{1}{7}\cot^7 x - \frac{1}{9}\cot^9 x + C}$

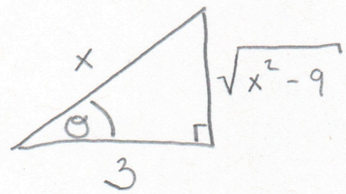
37 $\displaystyle\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$ LET $x = 3\sec\theta$

$dx = 3\tan\theta\sec\theta\, d\theta$

$$\longrightarrow \int \frac{\cancel{3}\tan\theta\ \cancel{\sec\theta}}{(3\sec\theta)^{\cancel{2}}\underbrace{\sqrt{(3\sec\theta)^2-9}}_{\sqrt{9(\sec^2\theta-1)}\ =\ \sqrt{9\tan^2\theta}\ =\ 3\tan\theta}}\,d\theta$$

$$= \int \frac{\sin\theta}{9\tan\theta}\,d\theta = \int \frac{1}{9}\cos\theta\,d\theta = \frac{1}{9}\sin\theta + C$$

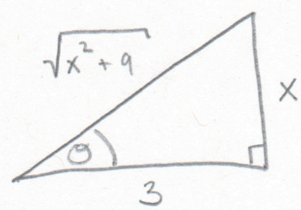
$\sec\theta = \dfrac{x}{3}$



$$\longrightarrow \boxed{\frac{\sqrt{x^2-9}}{9x} + C}$$

39 $\displaystyle\int \frac{x^3}{\sqrt{x^2+9}}\,dx$ LET $x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$



$$\longrightarrow \int \frac{(3\tan\theta)^3\, 3\sec^2\theta}{\sqrt{(3\tan\theta)^2+9}}\,d\theta = \int \frac{81\tan^3\theta\sec^2\theta}{3\sec\theta}\,d\theta$$

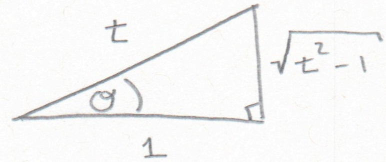
$$= 27\int \tan^3\theta\sec\theta\,d\theta = 27\left(\frac{1}{3}\sec^3 x - \sec x\right) + C$$

$$= 9\left(\frac{\sqrt{x^2+9}}{3}\right)^3 - 27\left(\frac{\sqrt{x^2+9}}{3}\right) + C$$

$$\boxed{= \frac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9} + C = \frac{1}{3}\sqrt{x^2+9}\left((x^2+9) - 27\right) + C = \frac{1}{3}\sqrt{x^2+9}\,(x^2-18) + C}$$

41 $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt$ LET $t = \sec\theta$

$dt = \tan\theta \sec\theta \, d\theta$



$t = \sqrt{2} \Rightarrow \sec\theta = \sqrt{2} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$t = 2 \Rightarrow \sec\theta = 2 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

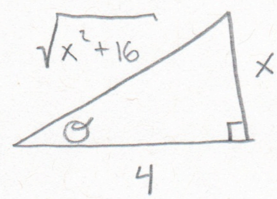
$$\longrightarrow \int_{\pi/4}^{\pi/3} \frac{\tan\theta \sec\theta}{\sec^3\theta\sqrt{\sec^2\theta - 1}}\,d\theta = \int_{\pi/4}^{\pi/3} \frac{1}{\sec^2\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \cos^2\theta\,d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/3} 1 + \cos(2\theta)\,d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{\pi/4}^{\pi/3}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{3} - \frac{\pi}{4}\right) + \frac{1}{2}\left(\frac{\sqrt{3}}{2} - 1\right)\right] = \frac{\pi}{24} + \frac{\sqrt{3}}{8} - \frac{1}{4} = \boxed{\frac{\pi + 3\sqrt{3} - 6}{24}}$$

45 $\displaystyle\int \frac{dx}{\sqrt{x^2+16}}$ LET $x = 4\tan\theta$

$dx = 4\sec^2\theta\,d\theta$



$$\longrightarrow \int \frac{4\sec^2\theta}{\sqrt{16\tan^2\theta + 16}}\,d\theta = \int \sec\theta\,d\theta = \ln|\tan\theta + \sec\theta| + C$$

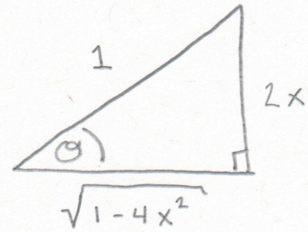
$$\longrightarrow \ln\left|\frac{x}{4} + \frac{\sqrt{x^2+16}}{4}\right| + C$$

$$= \ln|x + \sqrt{x^2+16}| \underbrace{- \ln(4) + C}_{\text{ARBITRARY}}$$

$$= \boxed{\ln|x + \sqrt{x^2+16}| + C}$$

47 $\int \sqrt{1-4x^2}\,dx$ LET $x = \frac{1}{2}\sin\theta$

$dx = \frac{1}{2}\cos\theta\,d\theta$



$$\longrightarrow \int \sqrt{1-4\left(\tfrac{1}{2}\sin\theta\right)^2}\;\tfrac{1}{2}\cos\theta\,d\theta$$

$$= \frac{1}{2}\int \cos^2\theta\,d\theta = \frac{1}{4}\int (1+\cos(2\theta))\,d\theta$$

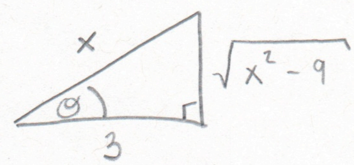
$$= \frac{1}{4}\theta + \frac{1}{8}\sin(2\theta) \longleftarrow \quad \sin 2\theta = 2\sin\theta\cos\theta$$

$$\longrightarrow = \frac{1}{4}\sin^{-1}(2x) + \frac{1}{8}\cdot 2(2x)\sqrt{1-4x^2}$$

$$= \boxed{\frac{1}{4}\sin^{-1}(2x) + \frac{1}{2}x\sqrt{1-4x^2} + C}$$

49 $\int \frac{\sqrt{x^2-9}}{x^3}\,dx$ LET $x = 3\sec\theta$

$dx = 3\tan\theta\sec\theta\,d\theta$



$$\longrightarrow \int \frac{3\tan\theta}{27\sec^3\theta}\cdot 3\tan\theta\sec\theta\,d\theta = \frac{1}{3}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$

$$= \frac{1}{3}\int \sin^2\theta\,d\theta = \frac{1}{6}\int 1-\cos(2\theta)\,d\theta$$

$$= \frac{1}{6}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C$$

$$= \frac{1}{6}\cos^{-1}\left(\frac{3}{x}\right) - \frac{1}{6}\left(\frac{\sqrt{x^2-9}}{x}\right)\left(\frac{3}{x}\right) + C$$

$$= \boxed{\frac{1}{6}\cos^{-1}\left(\frac{3}{x}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C}$$

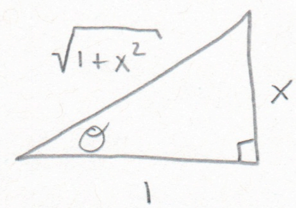
53 $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\,dx$ LET $u = x^2 - 7$

$du = 2x\,dx$

$$\longrightarrow \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\cdot 2\sqrt{u} \longrightarrow \boxed{\sqrt{x^2-7} + C}$$

55 $\displaystyle\int \frac{\sqrt{1+x^2}}{x}\,dx$ LET $x = \tan\theta$

$dx = \sec^2\theta\,d\theta$



$$\longrightarrow \int \frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta = \int \frac{\sec\theta}{\tan\theta}\left(\tan^2\theta + 1\right)d\theta$$

$$= \int \tan\theta\sec\theta\,d\theta + \int \csc\theta\,d\theta$$

SEE TABLE OF INTEGRALS IN BACK OF BOOK (OR EXERCISE # 33)

$$= \sec\theta + \ln\left|\csc\theta - \cot\theta\right| + C$$

$$\longrightarrow \sqrt{1+x^2} + \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + C$$

$$= \boxed{\sqrt{1+x^2} + \ln\left|\sqrt{1+x^2} - 1\right| - \ln|x| + C}$$