V EXAMPLE 8 Find the volume of a pyramid whose base is a square with side L and whose height is h.

SOLUTION We place the origin O at the vertex of the pyramid and the x-axis along its central axis as in Figure 14. Any plane P_x that passes through x and is perpendicular to the x-axis intersects the pyramid in a square with side of length s, say. We can express s in terms of x by observing from the similar triangles in Figure 15 that

$$\frac{x}{h} = \frac{s/2}{L/2} = \frac{s}{L}$$

and so $s = Lx/h$. [Another method is to observe that the line OP has slope $L/(2h)$ and so its equation is $y = Lx/(2h)$.] Thus the cross-sectional area is

$$A(x) = s^2 = \frac{L^2}{h^2}x^2$$

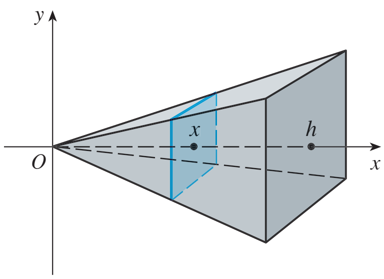


FIGURE 14

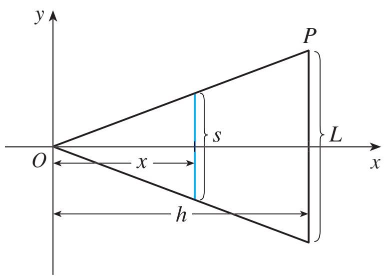


FIGURE 15

The pyramid lies between $x = 0$ and $x = h$, so its volume is

$$V = \int_0^h A(x)\,dx = \int_0^h \frac{L^2}{h^2}x^2\,dx = \frac{L^2}{h^2}\frac{x^3}{3}\bigg]_0^h = \frac{L^2h}{3}$$ ■

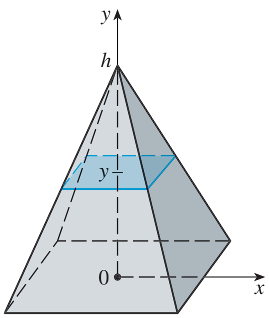


FIGURE 16

NOTE We didn't need to place the vertex of the pyramid at the origin in Example 8. We did so merely to make the equations simple. If, instead, we had placed the center of the base at the origin and the vertex on the positive y-axis, as in Figure 16, you can verify that we would have obtained the integral

$$V = \int_0^h \frac{L^2}{h^2}(h - y)^2\,dy = \frac{L^2h}{3}$$

7.2 EXERCISES

1–12 ▪ Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.

1. $y = 1/x,\ x = 1,\ x = 2,\ y = 0;$ about the x-axis

2. $y = 1 - x^2,\ y = 0;$ about the x-axis

3. $x = 2\sqrt{y},\ x = 0,\ y = 9;$ about the y-axis

4. $y = \ln x,\ y = 1,\ y = 2,\ x = 0;$ about the y-axis

5. $y = x^3, \; y = x, \; x \geqslant 0;$ about the x-axis

6. $y = \frac{1}{4}x^2, \; y = 5 - x^2;$ about the x-axis

7. $y^2 = x, \; x = 2y;$ about the y-axis

8. $y = x^{2/3}, \; x = 1, \; y = 0;$ about the y-axis

9. $y = x, \; y = \sqrt{x};$ about $y = 1$

10. $y = 1/x, \; y = 0, \; x = 1, \; x = 3;$ about $y = -1$

11. $y = x^2, \; x = y^2;$ about $x = -1$

12. $y = x, \; y = \sqrt{x};$ about $x = 2$

13. The region enclosed by the curves $y = x^3$ and $y = \sqrt{x}$ is rotated about the line $x = 1$. Find the volume of the resulting solid.

14. Find the volume of the solid obtained by rotating the region in Exercise 13 about the line $y = 1$.

15–16 ■ Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

15. $y = \tan^3 x, \; y = 1, \; x = 0;$ about $y = 1$

16. $y = (x - 2)^4, \; 8x - y = 16;$ about $x = 10$

17–18 ■ Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then find (approximately) the volume of the solid obtained by rotating about the x-axis the region bounded by these curves.

17. $y = x^2, \quad y = \ln(x + 1)$

18. $y = 3\sin(x^2), \quad y = e^{x/2} + e^{-2x}$

CAS **19–20** ■ Use a computer algebra system to find the exact volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

19. $y = \sin^2 x, \; y = 0, \; 0 \leqslant x \leqslant \pi;$ about $y = -1$

20. $y = x, \; y = xe^{1-x/2};$ about $y = 3$

21–22 ■ Each integral represents the volume of a solid. Describe the solid.

21. (a) $\pi \int_0^{\pi/2} \cos^2 x \, dx$ (b) $\pi \int_0^1 (y^4 - y^8) \, dy$

22. (a) $\pi \int_2^5 y \, dy$ (b) $\pi \int_0^{\pi/2} [(1 + \cos x)^2 - 1^2] \, dx$

23. A CAT scan produces equally spaced cross-sectional views of a human organ that provide information about the organ otherwise obtained only by surgery. Suppose that a CAT scan of a human liver shows cross-sections spaced 1.5 cm apart. The liver is 15 cm long and the cross-sectional areas, in square centimeters, are 0, 18, 58, 79, 94, 106, 117, 128, 63, 39, and 0. Use the Midpoint Rule to estimate the volume of the liver.

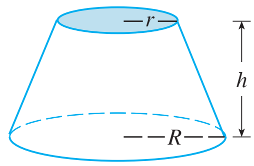
24. A log 10 m long is cut at 1-meter intervals and its cross-sectional areas A (at a distance x from the end of the log) are listed in the table. Use the Midpoint Rule with $n = 5$ to estimate the volume of the log.

x (m)	A (m^2)	x (m)	A (m^2)
0	0.68	6	0.53
1	0.65	7	0.55
2	0.64	8	0.52
3	0.61	9	0.50
4	0.58	10	0.48
5	0.59		

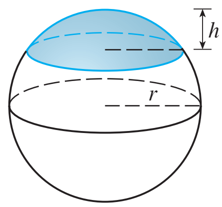
25–37 ■ Find the volume of the described solid S.

25. A right circular cone with height h and base radius r

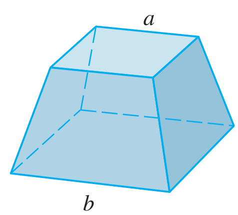
26. A frustum of a right circular cone with height h, lower base radius R, and top radius r



27. A cap of a sphere with radius r and height h



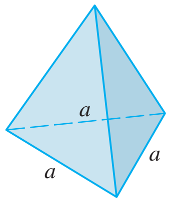
28. A frustum of a pyramid with square base of side b, square top of side a, and height h



What happens if $a = b$? What happens if $a = 0$?

29. A pyramid with height h and rectangular base with dimensions b and $2b$

30. A pyramid with height h and base an equilateral triangle with side a (a tetrahedron)



31. A tetrahedron with three mutually perpendicular faces and three mutually perpendicular edges with lengths 3 cm, 4 cm, and 5 cm

32. The base of S is a circular disk with radius r. Parallel cross-sections perpendicular to the base are squares.

33. The base of S is an elliptical region with boundary curve $9x^2 + 4y^2 = 36$. Cross-sections perpendicular to the x-axis are isosceles right triangles with hypotenuse in the base.

34. The base of S is the parabolic region $\{(x, y) \mid x^2 \leqslant y \leqslant 1\}$. Cross-sections perpendicular to the y-axis are equilateral triangles.

35. S has the same base as in Exercise 34, but cross-sections perpendicular to the y-axis are squares.

36. The base of S is the triangular region with vertices $(0, 0)$, $(3, 0)$, and $(0, 2)$. Cross-sections perpendicular to the y-axis are semicircles.

37. S has the same base as in Exercise 36, but cross-sections perpendicular to the y-axis are isosceles triangles with height equal to the base.

38. The base of S is a circular disk with radius r. Parallel cross-sections perpendicular to the base are isosceles triangles with height h and unequal side in the base.
(a) Set up an integral for the volume of S.
(b) By interpreting the integral as an area, find the volume of S.

39. Some of the pioneers of calculus, such as Kepler and Newton, were inspired by the problem of finding the volumes of wine barrels. (In fact Kepler published a book *Stereometria doliorum* in 1715 devoted to methods for finding the volumes of barrels.) They often approximated the shape of the sides by parabolas.
(a) A barrel with height h and maximum radius R is constructed by rotating about the x-axis the parabola $y = R - cx^2$, $-h/2 \leqslant x \leqslant h/2$, where c is a positive constant. Show that the radius of each end of the barrel is $r = R - d$, where $d = ch^2/4$.
(b) Show that the volume enclosed by the barrel is

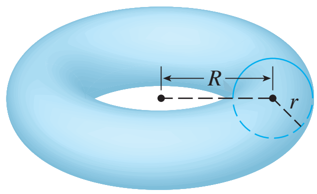
$$V = \tfrac{1}{3}\pi h\left(2R^2 + r^2 - \tfrac{2}{5}d^2\right)$$

CAS **40.** (a) A model for the shape of a bird's egg is obtained by rotating about the x-axis the region under the graph of

$$f(x) = (ax^3 + bx^2 + cx + d)\sqrt{1 - x^2}$$

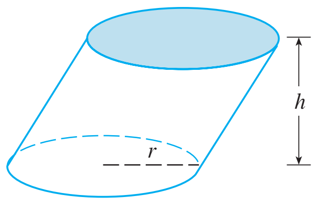
Use a CAS to find the volume of such an egg.
(b) For a Red-throated Loon, $a = -0.06$, $b = 0.04$, $c = 0.1$, and $d = 0.54$. Graph f and find the volume of an egg of this bird.

41. (a) Set up an integral for the volume of a solid *torus* (the donut-shaped solid shown in the figure) with radii r and R.
(b) By interpreting the integral as an area, find the volume of the torus.

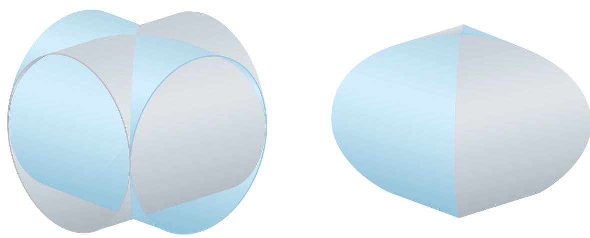


42. A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.

43. (a) Cavalieri's Principle states that if a family of parallel planes gives equal cross-sectional areas for two solids S_1 and S_2, then the volumes of S_1 and S_2 are equal. Prove this principle.
(b) Use Cavalieri's Principle to find the volume of the oblique cylinder shown in the figure.



44. Find the volume common to two circular cylinders, each with radius r, if the axes of the cylinders intersect at right angles.



45. Find the volume common to two spheres, each with radius r, if the center of each sphere lies on the surface of the other sphere.

46. A bowl is shaped like a hemisphere with diameter 30 cm. A ball with diameter 10 cm is placed in the bowl and water is poured into the bowl to a depth of h centimeters. Find the volume of water in the bowl.

47. A hole of radius r is bored through a cylinder of radius $R > r$ at right angles to the axis of the cylinder. Set up, but do not evaluate, an integral for the volume cut out.

48. A hole of radius r is bored through the center of a sphere of radius $R > r$. Find the volume of the remaining portion of the sphere.

7.3 VOLUMES BY CYLINDRICAL SHELLS

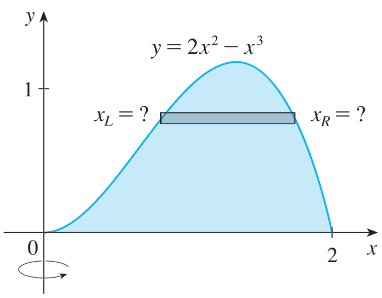


FIGURE 1

Some volume problems are very difficult to handle by the methods of the preceding section. For instance, let's consider the problem of finding the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$. (See Figure 1.) If we slice perpendicular to the y-axis, we get a washer. But to compute the inner radius and the outer radius of the washer, we would have to solve the cubic equation $y = 2x^2 - x^3$ for x in terms of y; that's not easy.

Fortunately, there is a method, called the **method of cylindrical shells**, that is easier to use in such a case. Figure 2 shows a cylindrical shell with inner radius r_1, outer radius r_2, and height h. Its volume V is calculated by subtracting the volume V_1 of the inner cylinder from the volume V_2 of the outer cylinder:

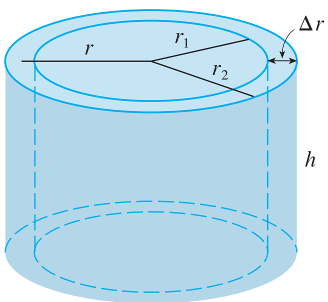


FIGURE 2

$$\begin{aligned} V &= V_2 - V_1 \\ &= \pi r_2^2 h - \pi r_1^2 h = \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1) \end{aligned}$$

If we let $\Delta r = r_2 - r_1$ (the thickness of the shell) and $r = \frac{1}{2}(r_2 + r_1)$ (the average radius of the shell), then this formula for the volume of a cylindrical shell becomes

1 $$\boxed{V = 2\pi rh\,\Delta r}$$

and it can be remembered as

$$V = [\text{circumference}][\text{height}][\text{thickness}]$$

Now let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x)$ [where f is continuous and $f(x) \geqslant 0$], $y = 0$, $x = a$, and $x = b$, where $b > a \geqslant 0$. (See Figure 3.)

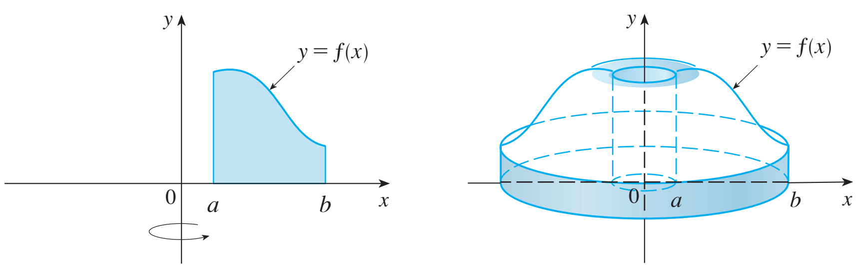


FIGURE 3