V EXAMPLE 7 Evaluate $\int x^2 e^{x^3} dx$.

SOLUTION We substitute $u = x^3$. Then $du = 3x^2 dx$, so $x^2 dx = \frac{1}{3} du$ and

$$\int x^2 e^{x^3} dx = \frac{1}{3} \int e^u du = \frac{1}{3} e^u + C = \frac{1}{3} e^{x^3} + C$$

EXAMPLE 8 Find the area under the curve $y = e^{-3x}$ from 0 to 1.

SOLUTION The area is

$$A = \int_0^1 e^{-3x} dx = -\frac{1}{3} e^{-3x} \Big]_0^1 = \frac{1}{3} (1 - e^{-3})$$

5.3 EXERCISES

- **I.** (a) How is the number e defined?
 - (b) What is an approximate value for e?
 - (c) Sketch, by hand, the graph of the function $f(x) = e^x$ with particular attention to how the graph crosses the y-axis. What fact allows you to do this?
- **2–4** Simplify each expression.
- **2.** (a) $e^{\ln 6}$

- (b) $\ln \sqrt{e}$
- **3.** (a) $\ln e^{\sqrt{2}}$
- (b) $e^{3 \ln 2}$
- **4.** (a) $\ln e^{\sin x}$
- (b) $a^{x} + \ln x$

5−8 • Solve each equation for x.

- **5.** (a) $2 \ln x = 1$
- (b) $e^{-x} = 5$
- **6.** (a) $e^{2x+3} 7 = 0$
- (b) ln(5-2x) = -3
- 7. (a) $2^{x-5} = 3$
- (b) $\ln x + \ln(x 1) = 1$
- **8.** (a) $\ln(\ln x) = 1$
- (b) $e^{ax} = Ce^{bx}$, where $a \neq b$

9–10 • Solve each inequality for x.

- **9.** (a) $e^x < 10$
- (b) $\ln x > -1$
- **10.** (a) $2 < \ln x < 9$
- (b) $e^{2-3x} > 4$

11–14 • Make a rough sketch of the graph of each function. Do not use a calculator. Just use the graph given in Figure 2 and, if necessary, the transformations of Section 1.3.

11.
$$y = e^{-x}$$

12.
$$y = 1 + 2e^x$$

13.
$$y = 3 - e^x$$

14.
$$y = 2 + 5(1 - e^{-x})$$

.

15–20 ■ Find the limit.

15. $\lim_{x \to 0} e^{1-x^3}$

- **16.** $\lim_{x \to (\pi/2)^+} e^{\tan x}$
- 17. $\lim_{x\to\infty} \frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}$
- **18.** $\lim_{x \to -\infty} \frac{e^{3x} e^{-3x}}{e^{3x} + e^{-3x}}$
- **19.** $\lim_{x\to 2^+} e^{3/(2-x)}$
- **20.** $\lim_{x \to 2^{-}} e^{3/(2-x)}$

21-34 Differentiate the function.

- **21.** $f(x) = x^2 e^x$
- **22.** $y = \frac{e^x}{1+x}$

23. $y = e^{ax^3}$

- **24.** $y = e^{u}(\cos u + cu)$
- **25.** $f(u) = e^{1/u}$
- **26.** $y = e^x \ln x$
- $27. F(t) = e^{t \sin 2t}$
- $28. \ y = e^{k \tan \sqrt{x}}$
- **29.** $y = \sqrt{1 + 2e^{3x}}$
- **30.** $y = \cos(e^{\pi x})$

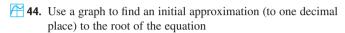
31. $y = e^{e^x}$

- **32.** $y = \sqrt{1 + xe^{-2x}}$
- $33. \ y = \frac{ae^x + b}{ce^x + d}$
- **34.** $y = \frac{e^x + e^{-x}}{e^x e^{-x}}$

35–36 • Find an equation of the tangent line to the curve at the given point.

- **35.** $y = e^{2x} \cos \pi x$, (0, 1)
- **36.** $y = e^x/x$, $(1, e^x)$
- **37.** Find y' if $e^{x^2y} = x + y$.
- **38.** Show that the function $y = Ae^{-x} + Bxe^{-x}$ satisfies the differential equation y'' + 2y' + y = 0.

- **40.** Find the values of λ for which $y = e^{\lambda x}$ satisfies the equation y + y' = y''.
- **41.** If $f(x) = e^{2x}$, find a formula for $f^{(n)}(x)$.
- **42.** Find the thousandth derivative of $f(x) = xe^{-x}$.
- **43.** (a) Use the Intermediate Value Theorem to show that there is a root of the equation $e^x + x = 0$.
 - (b) Use Newton's method to find the root of the equation in part (a) correct to six decimal places.



$$e^{-x^2} = x^3 + x - 3$$

Then use Newton's method to find the root correct to six decimal places.

45. Under certain circumstances a rumor spreads according to the equation

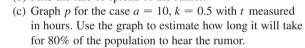
$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where p(t) is the proportion of the population that knows the rumor at time t and a and k are positive constants.

(a) Find $\lim_{t\to\infty} p(t)$.

M

(b) Find the rate of spread of the rumor.



- **46.** An object is attached to the end of a vibrating spring and its displacement from its equilibrium position is $y = 8e^{-t/2} \sin 4t$, where t is measured in seconds and y is measured in centimeters.
 - (a) Graph the displacement function together with the functions $y = 8e^{-t/2}$ and $y = -8e^{-t/2}$. How are these graphs related? Can you explain why?
 - (b) Use the graph to estimate the maximum value of the displacement. Does it occur when the graph touches the graph of $y = 8e^{-t/2}$?
 - (c) What is the velocity of the object when it first returns to its equilibrium position?
 - (d) Use the graph to estimate the time after which the displacement is no more than 2 cm from equilibrium.
 - 47. Find the absolute maximum value of the function $f(x) = x - e^x$.
 - **48.** Find the absolute minimum value of the function $q(x) = e^{x}/x, x > 0.$
 - **49.** On what interval is the curve $y = xe^{3x}$ concave upward?
 - **50.** On what interval is the function $f(x) = x^2 e^{-x}$ increasing?

51-53 • Discuss the curve using the guidelines of Section 3.4.

51.
$$v = 1/(1 + e^{-x})$$

52.
$$v = e^{2x} - e^x$$

53.
$$y = e^{3x} + e^{-2x}$$

f 54-55 • Draw a graph of f that shows all the important aspects of the curve. Estimate the local maximum and minimum values and then use calculus to find these values exactly. Use a graph of f'' to estimate the inflection points.

54.
$$f(x) = e^{\cos x}$$

55.
$$f(x) = e^{x^3 - x}$$

56. The family of bell-shaped curves

$$y = \frac{1}{\sigma \sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

occurs in probability and statistics, where it is called the normal density function. The constant μ is called the mean and the positive constant σ is called the *standard deviation*. For simplicity, let's scale the function so as to remove the factor $1/(\sigma\sqrt{2\pi})$ and let's analyze the special case where $\mu = 0$. So we study the function

$$f(x) = e^{-x^2/(2\sigma^2)}$$

- (a) Find the asymptote, maximum value, and inflection points of f.
- (b) What role does σ play in the shape of the curve?



(c) Illustrate by graphing four members of this family on the same screen.

57–64 • Evaluate the integral.

57.
$$\int_{0}^{5} e^{-3x} dx$$

58.
$$\int_{0}^{1} xe^{-x^{2}} dx$$

$$59. \int e^x \sqrt{1 + e^x} \, dx$$

$$\mathbf{60.} \int \sec^2 x \, e^{\tan x} \, dx$$

$$\textbf{61.} \int \frac{e^x + 1}{e^x} \, dx$$

$$62. \int \frac{e^{1/x}}{x^2} dx$$

$$63. \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

$$64. \int e^x \sin(e^x) \, dx$$

65-66 Find the inverse function of f. Check your answer by graphing both f and f^{-1} on the same screen.

65.
$$f(x) = \ln(x+3)$$
 66. $f(x) = \frac{1+e^x}{1-e^x}$

66.
$$f(x) = \frac{1 + e^x}{1 - e^x}$$

67. If
$$f(x) = 3 + x + e^x$$
, find $(f^{-1})'(4)$.

68. Evaluate
$$\lim_{x \to \pi} \frac{e^{\sin x} - 1}{x - \pi}$$
.

- **69.** Prove the second law of exponents [see (7)].
- **70.** Prove the third law of exponents [see (7)].
- **71.** (a) Show that $e^x \ge 1 + x$ if $x \ge 0$. [Hint: Show that $f(x) = e^x - (1 + x)$ is increasing for x > 0.
 - (b) Deduce that $\frac{4}{3} \le \int_0^1 e^{x^2} dx \le e$.
- **72.** (a) Use the inequality of Exercise 71(a) to show that, for $x \ge 0$,

$$e^x \ge 1 + x + \frac{1}{2}x^2$$

- (b) Use part (a) to improve the estimate of $\int_0^1 e^{x^2} dx$ given in Exercise 71(b).
- **73.** (a) Use mathematical induction to prove that for $x \ge 0$ and any positive integer n,

$$e^x \ge 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

- (b) Use part (a) to show that e > 2.7.
- (c) Use part (a) to show that

$$\lim_{x\to\infty}\frac{e^x}{x^k}=\infty$$

for any positive integer k.

74. This exercise illustrates Exercise 73(c) for the case k = 10.

- (a) Compare the rates of growth of $f(x) = x^{10}$ and $q(x) = e^x$ by graphing both f and q in several viewing rectangles. When does the graph of q finally surpass the
- (b) Find a viewing rectangle that shows how the function $h(x) = e^{x}/x^{10}$ behaves for large x.
- (c) Find a number N such that

$$\frac{e^x}{x^{10}} > 10^{10} \qquad \text{whenever} \qquad x > N$$

5.4 GENERAL LOGARITHMIC AND EXPONENTIAL FUNCTIONS

In this section we use the natural exponential and logarithmic functions to study exponential and logarithmic functions with base a > 0.

GENERAL EXPONENTIAL FUNCTIONS

If a > 0 and r is any rational number, then by (4) and (7) in Section 5.3,

$$a^{r} = (e^{\ln a})^{r} = e^{r \ln a}$$

Therefore, even for irrational numbers x, we define



$$a^x = e^{x \ln a}$$

Thus, for instance,

$$2^{\sqrt{3}} = e^{\sqrt{3} \ln 2} \approx e^{1.20} \approx 3.32$$

The function $f(x) = a^x$ is called the **exponential function with base a**. Notice that a^x is positive for all x because e^x is positive for all x.

Definition 1 allows us to extend one of the laws of logarithms. We know that $\ln(a^r) = r \ln a$ when r is rational. But if we now let r be any real number we have, from Definition 1,

$$\ln a^r = \ln(e^{r \ln a}) = r \ln a$$

Thus



 $\ln a^r = r \ln a$ for any real number r

The general laws of exponents follow from Definition 1 together with the laws of exponents for e^x .