NOTE 1 Replacing a by the general number x in the formula of Theorem 7, we get

$$\boxed{8} \qquad (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

If we write $y = f^{-1}(x)$, then $f(y) = x$, so Equation 8, when expressed in Leibniz notation, becomes

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}$$

NOTE 2 If it is known in advance that f^{-1} is differentiable, then its derivative can be computed more easily than in the proof of Theorem 7 by using implicit differentiation. If $y = f^{-1}(x)$, then $f(y) = x$. Differentiating the equation $f(y) = x$ implicitly with respect to x, remembering that y is a function of x, and using the Chain Rule, we get

$$f'(y)\,\frac{dy}{dx} = 1$$

Therefore

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{\dfrac{dx}{dy}}$$

V EXAMPLE 6 If $f(x) = 2x + \cos x$, find $(f^{-1})'(1)$.

SOLUTION Notice that f is one-to-one because

$$f'(x) = 2 - \sin x > 0$$

and so f is increasing. To use Theorem 7 we need to know $f^{-1}(1)$ and we can find it by inspection:

$$f(0) = 1 \quad \Rightarrow \quad f^{-1}(1) = 0$$

Therefore

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{2 - \sin 0} = \frac{1}{2}$$ ▪

5.1 EXERCISES

1. (a) What is a one-to-one function?
(b) How can you tell from the graph of a function whether it is one-to-one?

2. (a) Suppose f is a one-to-one function with domain A and range B. How is the inverse function f^{-1} defined? What is the domain of f^{-1}? What is the range of f^{-1}?
(b) If you are given a formula for f, how do you find a formula for f^{-1}?
(c) If you are given the graph of f, how do you find the graph of f^{-1}?

3–14 ▪ A function is given by a table of values, a graph, a formula, or a verbal description. Determine whether it is one-to-one.

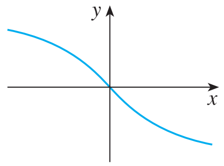
3.

x	1	2	3	4	5	6
$f(x)$	1.5	2.0	3.6	5.3	2.8	2.0

4.

x	1	2	3	4	5	6
$f(x)$	1	2	4	8	16	32

5.



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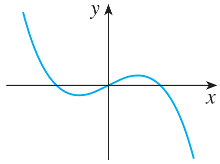

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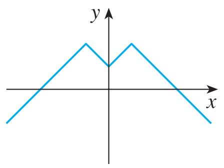

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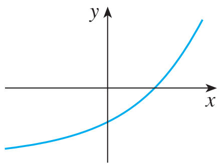


9. $f(x) = x^2 - 2x$

10. $f(x) = 10 - 3x$

11. $g(x) = 1/x$

12. $g(x) = \cos x$

13. $f(t)$ is the height of a football t seconds after kickoff.

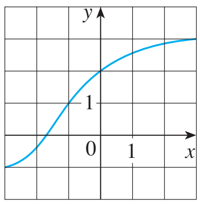
14. $f(t)$ is your height at age t.

15. If f is a one-to-one function such that $f(2) = 9$, what is $f^{-1}(9)$?

16. If $f(x) = x + \cos x$, find $f^{-1}(1)$.

17. If $h(x) = x + \sqrt{x}$, find $h^{-1}(6)$.

18. The graph of f is given.
 (a) Why is f one-to-one?
 (b) What are the domain and range of f^{-1}?
 (c) What is the value of $f^{-1}(2)$?
 (d) Estimate the value of $f^{-1}(0)$.



19. The formula $C = \frac{5}{9}(F - 32)$, where $F \geqslant -459.67$, expresses the Celsius temperature C as a function of the Fahrenheit temperature F. Find a formula for the inverse function and interpret it. What is the domain of the inverse function?

20. In the theory of relativity, the mass of a particle with speed v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and c is the speed of light in a vacuum. Find the inverse function of f and explain its meaning.

21–26 ▪ Find a formula for the inverse of the function.

21. $f(x) = 3 - 2x$

22. $f(x) = \dfrac{4x - 1}{2x + 3}$

23. $f(x) = \sqrt{10 - 3x}$

24. $y = 2x^3 + 3$

25. $y = \dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}$

26. $f(x) = 2x^2 - 8x, \quad x \geqslant 2$

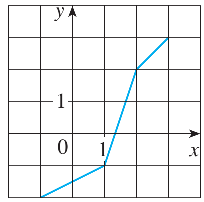
27–28 ▪ Find an explicit formula for f^{-1} and use it to graph f^{-1}, f, and the line $y = x$ on the same screen. To check your work, see whether the graphs of f and f^{-1} are reflections about the line.

27. $f(x) = x^4 + 1, \quad x \geqslant 0$

28. $f(x) = \sqrt{x^2 + 2x}, \quad x > 0$

29–30 ▪ Use the given graph of f to sketch the graph of f^{-1}.

29.



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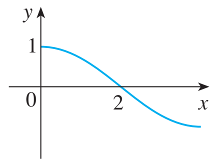

31–34 ▪
(a) Show that f is one-to-one.
(b) Use Theorem 7 to find $(f^{-1})'(a)$.
(c) Calculate $f^{-1}(x)$ and state the domain and range of f^{-1}.
(d) Calculate $(f^{-1})'(a)$ from the formula in part (c) and check that it agrees with the result of part (b).
(e) Sketch the graphs of f and f^{-1} on the same axes.

31. $f(x) = x^3, \quad a = 8$

32. $f(x) = \sqrt{x - 2}, \quad a = 2$

33. $f(x) = 9 - x^2, \quad 0 \leqslant x \leqslant 3, \quad a = 8$

34. $f(x) = 1/(x - 1), \quad x > 1, \quad a = 2$

35–38 ▪ Find $(f^{-1})'(a)$.

35. $f(x) = x^3 + x + 1, \quad a = 1$

36. $f(x) = x^5 - x^3 + 2x, \quad a = 2$

37. $f(x) = 3 + x^2 + \tan(\pi x/2), \quad -1 < x < 1, \quad a = 3$

38. $f(x) = \sqrt{x^3 + x^2 + x + 1}, \quad a = 2$

39. Suppose f^{-1} is the inverse function of a differentiable function f and $f(4) = 5$, $f'(4) = \frac{2}{3}$. Find $(f^{-1})'(5)$.

40. Suppose f^{-1} is the inverse function of a differentiable function f and let $G(x) = 1/f^{-1}(x)$. If $f(3) = 2$ and $f'(3) = \frac{1}{9}$, find $G'(2)$.

41. Show that the function $f(x) = \sqrt{x^3 + x^2 + x + 1}$ is one-to-one. Use a computer algebra system to find an explicit expression for $f^{-1}(x)$. (Your CAS will produce three possible expressions. Explain why two of them are irrelevant in this context.)

42. Show that $h(x) = \sin x$, $x \in \mathbb{R}$, is not one-to-one, but its restriction $f(x) = \sin x$, $-\pi/2 \leqslant x \leqslant \pi/2$, is one-to-one. Compute the derivative of $f^{-1} = \sin^{-1}$ by the method of Note 2.

43. (a) If we shift a curve to the left, what happens to its reflection about the line $y = x$? In view of this geometric principle, find an expression for the inverse of $g(x) = f(x + c)$, where f is a one-to-one function.

(b) Find an expression for the inverse of $h(x) = f(cx)$, where $c \neq 0$.

44. (a) If f is a one-to-one, twice differentiable function with inverse function g, show that

$$g''(x) = -\frac{f''(g(x))}{[f'(g(x))]^3}$$

(b) Deduce that if f is increasing and concave upward, then its inverse function is concave downward.

5.2 THE NATURAL LOGARITHMIC FUNCTION

In this section we define the natural logarithm as an integral and then show that it obeys the usual laws of logarithms. The Fundamental Theorem makes it easy to differentiate this function.

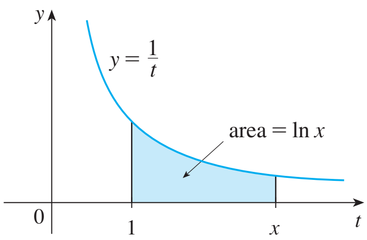

FIGURE 1

1 DEFINITION The **natural logarithmic function** is the function defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad x > 0$$

The existence of this function depends on the fact that the integral of a continuous function always exists. If $x > 1$, then $\ln x$ can be interpreted geometrically as the area under the hyperbola $y = 1/t$ from $t = 1$ to $t = x$. (See Figure 1.) For $x = 1$, we have

$$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$$

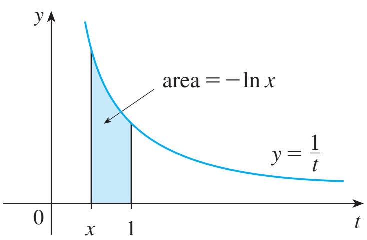

FIGURE 2

For $0 < x < 1$,
$$\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt < 0$$

and so $\ln x$ is the negative of the area shown in Figure 2.

V EXAMPLE 1

(a) By comparing areas, show that $\frac{1}{2} < \ln 2 < \frac{3}{4}$.

(b) Use the Midpoint Rule with $n = 10$ to estimate the value of $\ln 2$.

SOLUTION

(a) We can interpret $\ln 2$ as the area under the curve $y = 1/t$ from 1 to 2. From Figure 3 we see that this area is larger than the area of rectangle $BCDE$ and smaller than the area of trapezoid $ABCD$. Thus we have

$$\tfrac{1}{2} \cdot 1 < \ln 2 < 1 \cdot \tfrac{1}{2}\left(1 + \tfrac{1}{2}\right)$$

$$\tfrac{1}{2} < \ln 2 < \tfrac{3}{4}$$

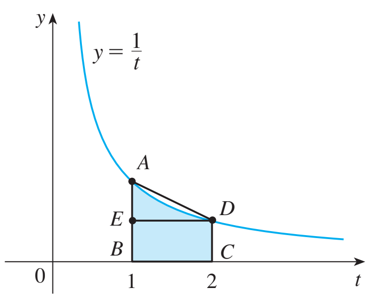

FIGURE 3