57. $y = 3 \sec \pi(x + \frac{1}{2})$

58. $y = \sec\left(3x + \dfrac{\pi}{2}\right)$

59. $y = -2 \tan\left(2x - \dfrac{\pi}{3}\right)$

60. $y = 2 \cot(3\pi x + 3\pi)$

APPLICATIONS

61. Lighthouse The beam from a lighthouse completes one rotation every 2 min. At time t, the distance d shown in the figure below is

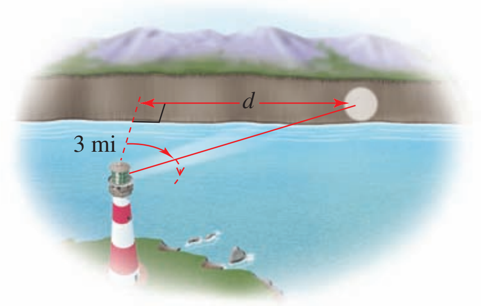
$$d(t) = 3 \tan \pi t$$

where t is measured in minutes and d in miles.

(a) Find $d(0.15)$, $d(0.25)$, and $d(0.45)$.

(b) Sketch a graph of the function d for $0 \le t < \frac{1}{2}$.

(c) What happens to the distance d as t approaches $\frac{1}{2}$?



62. Length of a Shadow On a day when the sun passes directly overhead at noon, a 6-ft-tall man casts a shadow of length

$$S(t) = 6\left|\cot \frac{\pi}{12} t\right|$$

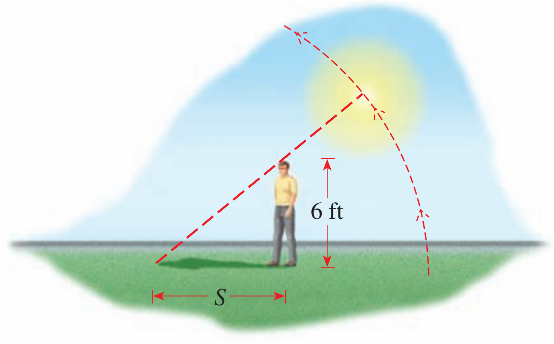
where S is measured in feet and t is the number of hours since 6 A.M.

(a) Find the length of the shadow at 8:00 A.M., noon, 2:00 P.M., and 5:45 P.M.

(b) Sketch a graph of the function S for $0 < t < 12$.

(c) From the graph, determine the values of t at which the length of the shadow equals the man's height. To what time of day does each of these values correspond?

(d) Explain what happens to the shadow as the time approaches 6 P.M. (that is, as $t \to 12^-$).



DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

63. PROVE: Periodic Functions **(a)** Prove that if f is periodic with period p, then $1/f$ is also periodic with period p.

(b) Prove that cosecant and secant both have period 2π.

64. PROVE: Periodic Functions Prove that if f and g are periodic with period p, then f/g is also periodic but its period could be smaller than p.

65. PROVE: Reduction Formulas Use the graphs in Figure 5 to explain why the following formulas are true.

$$\tan\left(x - \frac{\pi}{2}\right) = -\cot x \qquad \sec\left(x - \frac{\pi}{2}\right) = \csc x$$

5.5 INVERSE TRIGONOMETRIC FUNCTIONS AND THEIR GRAPHS

■ The Inverse Sine Function ■ The Inverse Cosine Function ■ The Inverse Tangent Function ■ The Inverse Secant, Cosecant, and Cotangent Functions

We study applications of inverse trigonometric functions to triangles in Sections 6.4–6.6.

Recall from Section 2.8 that the inverse of a function f is a function f^{-1} that reverses the rule of f. For a function to have an inverse, it must be one-to-one. Since the trigonometric functions are not one-to-one, they do not have inverses. It is possible, however, to restrict the domains of the trigonometric functions in such a way that the resulting functions are one-to-one.

■ The Inverse Sine Function

Let's first consider the sine function. There are many ways to restrict the domain of sine so that the new function is one-to-one. A natural way to do this is to restrict the domain to the interval $[-\pi/2, \pi/2]$. The reason for this choice is that sine is one-to-one on this interval and moreover attains each of the values in its range on this interval. From Figure 1 we see that sine is one-to-one on this restricted domain (by the Horizontal Line Test) and so has an inverse.

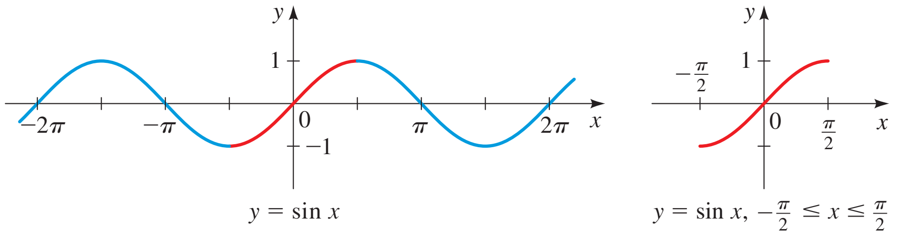


FIGURE 1 Graphs of the sine function and the restricted sine function

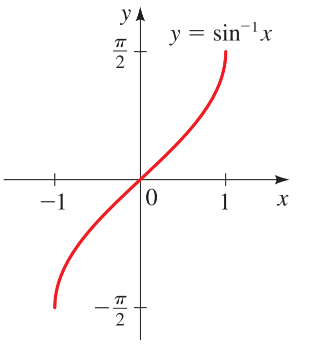


FIGURE 2 Graph of $y = \sin^{-1} x$

We can now define an inverse sine function on this restricted domain. The graph of $y = \sin^{-1} x$ is shown in Figure 2; it is obtained by reflecting the graph of $y = \sin x$, $-\pi/2 \le x \le \pi/2$, in the line $y = x$.

DEFINITION OF THE INVERSE SINE FUNCTION

The **inverse sine function** is the function $\sin^{-1}$ with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$ defined by

$$\sin^{-1} x = y \quad \Leftrightarrow \quad \sin y = x$$

The inverse sine function is also called **arcsine**, denoted by **arcsin**.

Thus $y = \sin^{-1} x$ *is the number in the interval* $[-\pi/2, \pi/2]$ *whose sine is* x. In other words, $\sin(\sin^{-1} x) = x$. In fact, from the general properties of inverse functions studied in Section 2.8, we have the following **cancellation properties**.

$$\sin(\sin^{-1} x) = x \quad \text{for} \quad -1 \le x \le 1$$
$$\sin^{-1}(\sin x) = x \quad \text{for} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

EXAMPLE 1 ■ Evaluating the Inverse Sine Function

Find each value.

(a) $\sin^{-1} \frac{1}{2}$ **(b)** $\sin^{-1}\left(-\frac{1}{2}\right)$ **(c)** $\sin^{-1} \frac{3}{2}$

SOLUTION

(a) The number in the interval $[-\pi/2, \pi/2]$ whose sine is $\frac{1}{2}$ is $\pi/6$. Thus $\sin^{-1} \frac{1}{2} = \pi/6$.

(b) The number in the interval $[-\pi/2, \pi/2]$ whose sine is $-\frac{1}{2}$ is $-\pi/6$. Thus $\sin^{-1}\left(-\frac{1}{2}\right) = -\pi/6$.

(c) Since $\frac{3}{2} > 1$, it is not in the domain of $\sin^{-1} x$, so $\sin^{-1} \frac{3}{2}$ is not defined.

Now Try Exercise 3 ■

EXAMPLE 2 ■ Using a Calculator to Evaluate Inverse Sine

Find approximate values for **(a)** $\sin^{-1}(0.82)$ and **(b)** $\sin^{-1}\frac{1}{3}$.

SOLUTION

We use a calculator to approximate these values. Using the $\boxed{\text{SIN}^{-1}}$, or $\boxed{\text{INV}}$ $\boxed{\text{SIN}}$, or $\boxed{\text{ARC}}$ $\boxed{\text{SIN}}$ key(s) on the calculator (with the calculator in radian mode), we get

(a) $\sin^{-1}(0.82) \approx 0.96141$ **(b)** $\sin^{-1}\frac{1}{3} \approx 0.33984$

Now Try Exercises 11 and 21 ■

When evaluating expressions involving $\sin^{-1}$, we need to remember that the range of $\sin^{-1}$ is the interval $[-\pi/2, \pi/2]$.

EXAMPLE 3 ■ Evaluating Expressions with Inverse Sine

Find each value.

(a) $\sin^{-1}\left(\sin\dfrac{\pi}{3}\right)$ **(b)** $\sin^{-1}\left(\sin\dfrac{2\pi}{3}\right)$

SOLUTION

(a) Since $\pi/3$ is in the interval $[-\pi/2, \pi/2]$, we can use the cancellation properties of inverse functions (page 440):

$$\sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3} \qquad \text{Cancellation property: } -\frac{\pi}{2} \le \frac{\pi}{3} \le \frac{\pi}{2}$$

(b) We first evaluate the expression in the parentheses:

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \qquad \text{Evaluate}$$

$$= \frac{\pi}{3} \qquad \text{Because } \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Note: $\sin^{-1}(\sin x) = x$ only if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

Now Try Exercises 31 and 33 ■

■ The Inverse Cosine Function

If the domain of the cosine function is restricted to the interval $[0, \pi]$, the resulting function is one-to-one and so has an inverse. We choose this interval because on it, cosine attains each of its values exactly once (see Figure 3).

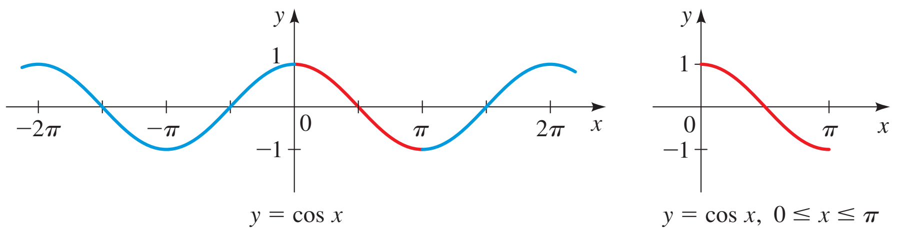


FIGURE 3 Graphs of the cosine function and the restricted cosine function

DEFINITION OF THE INVERSE COSINE FUNCTION

The **inverse cosine function** is the function $\cos^{-1}$ with domain $[-1, 1]$ and range $[0, \pi]$ defined by

$$\cos^{-1} x = y \quad \Leftrightarrow \quad \cos y = x$$

The inverse cosine function is also called **arccosine**, denoted by **arccos**.

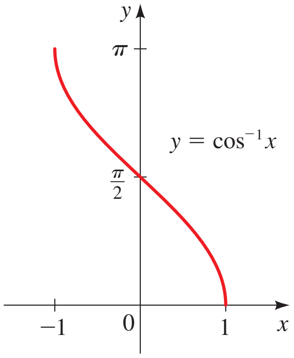


FIGURE 4 Graph of $y = \cos^{-1} x$

Thus $y = \cos^{-1} x$ *is the number in the interval* $[0, \pi]$ *whose cosine is* x. The following **cancellation properties** follow from the inverse function properties.

$$\boxed{\begin{aligned} \cos(\cos^{-1} x) &= x \quad \text{for} \quad -1 \le x \le 1 \\ \cos^{-1}(\cos x) &= x \quad \text{for} \quad 0 \le x \le \pi \end{aligned}}$$

The graph of $y = \cos^{-1} x$ is shown in Figure 4; it is obtained by reflecting the graph of $y = \cos x$, $0 \le x \le \pi$, in the line $y = x$.

EXAMPLE 4 ■ Evaluating the Inverse Cosine Function

Find each value.

(a) $\cos^{-1} \dfrac{\sqrt{3}}{2}$ **(b)** $\cos^{-1} 0$ **(c)** $\cos^{-1}\left(-\dfrac{1}{2}\right)$

SOLUTION

(a) The number in the interval $[0, \pi]$ whose cosine is $\sqrt{3}/2$ is $\pi/6$. Thus $\cos^{-1}(\sqrt{3}/2) = \pi/6$.

(b) The number in the interval $[0, \pi]$ whose cosine is 0 is $\pi/2$. Thus $\cos^{-1} 0 = \pi/2$.

(c) The number in the interval $[0, \pi]$ whose cosine is $-\frac{1}{2}$ is $2\pi/3$. Thus $\cos^{-1}(-\frac{1}{2}) = 2\pi/3$. (The graph in Figure 4 shows that if $-1 \le x < 0$, then $\cos^{-1} x > \pi/2$.)

Now Try Exercises 5 and 13 ■

EXAMPLE 5 ■ Evaluating Expressions with Inverse Cosine

Find each value.

(a) $\cos^{-1}\left(\cos \dfrac{2\pi}{3}\right)$ **(b)** $\cos^{-1}\left(\cos \dfrac{5\pi}{3}\right)$

SOLUTION

(a) Since $2\pi/3$ is in the interval $[0, \pi]$ we can use the above cancellation properties:

$$\cos^{-1}\left(\cos \frac{2\pi}{3}\right) = \frac{2\pi}{3} \qquad \text{Cancellation property: } 0 \le \frac{2\pi}{3} \le \pi$$

(b) We first evaluate the expression in the parentheses:

$$\begin{aligned} \cos^{-1}\left(\cos \frac{5\pi}{3}\right) &= \cos^{-1}\left(\frac{1}{2}\right) && \text{Evaluate} \\ &= \frac{\pi}{3} && \text{Because } \cos \frac{\pi}{3} = \frac{1}{2} \end{aligned}$$

Note: $\cos^{-1}(\cos x) = x$ only if $0 \le x \le \pi$.

Now Try Exercises 35 and 37 ■

■ The Inverse Tangent Function

We restrict the domain of the tangent function to the interval $(-\pi/2, \pi/2)$ to obtain a one-to-one function.

DEFINITION OF THE INVERSE TANGENT FUNCTION

The **inverse tangent function** is the function $\tan^{-1}$ with domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$ defined by

$$\tan^{-1} x = y \quad \Leftrightarrow \quad \tan y = x$$

The inverse tangent function is also called **arctangent**, denoted by **arctan**.

Thus $y = \tan^{-1} x$ *is the number in the interval* $(-\pi/2, \pi/2)$ *whose tangent is* x. The following **cancellation properties** follow from the inverse function properties.

$$\tan(\tan^{-1} x) = x \quad \text{for} \quad x \in \mathbb{R}$$
$$\tan^{-1}(\tan x) = x \quad \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Figure 5 shows the graph of $y = \tan x$ on the interval $(-\pi/2, \pi/2)$ and the graph of its inverse function, $y = \tan^{-1} x$.

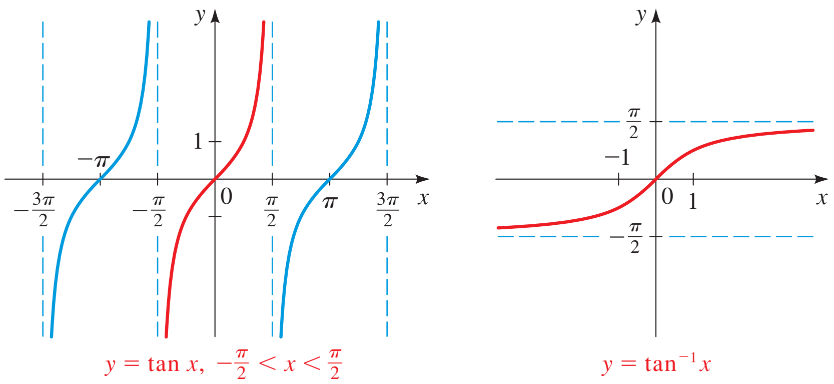


FIGURE 5 Graphs of the restricted tangent function and the inverse tangent function

EXAMPLE 6 ■ Evaluating the Inverse Tangent Function

Find each value.

(a) $\tan^{-1} 1$ **(b)** $\tan^{-1} \sqrt{3}$ **(c)** $\tan^{-1}(20)$

SOLUTION

(a) The number in the interval $(-\pi/2, \pi/2)$ with tangent 1 is $\pi/4$. Thus $\tan^{-1} 1 = \pi/4$.

(b) The number in the interval $(-\pi/2, \pi/2)$ with tangent $\sqrt{3}$ is $\pi/3$. Thus $\tan^{-1} \sqrt{3} = \pi/3$.

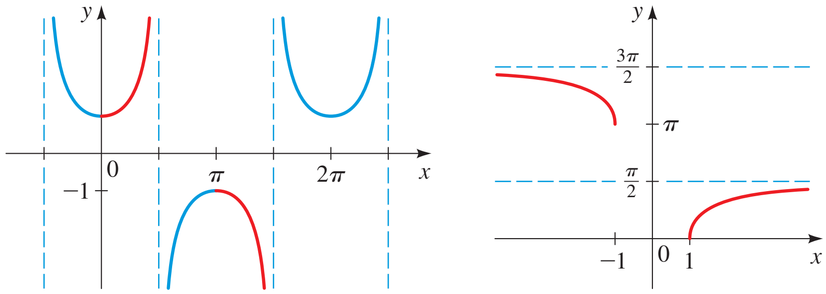
(c) We use a calculator (in radian mode) to find that $\tan^{-1}(20) \approx 1.52084$.

Now Try Exercises 7 and 17

■ The Inverse Secant, Cosecant, and Cotangent Functions

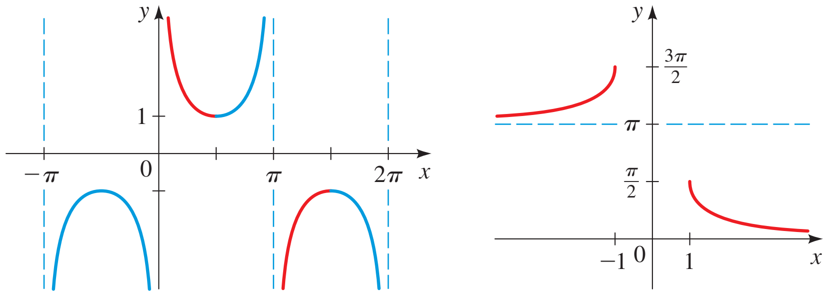
See Exercise 46 in Section 6.4 (page 508) for a way of finding the values of these inverse trigonometric functions on a calculator.

To define the inverse functions of the secant, cosecant, and cotangent functions, we restrict the domain of each function to a set on which it is one-to-one and on which it attains all its values. Although any interval satisfying these criteria is appropriate, we choose to restrict the domains in a way that simplifies the choice of sign in computations involving inverse trigonometric functions. The choices we make are also appropriate for calculus. This explains the seemingly strange restriction for the domains of the secant and cosecant functions. We end this section by displaying the graphs of the secant, cosecant, and cotangent functions with their restricted domains and the graphs of their inverse functions (Figures 6–8).



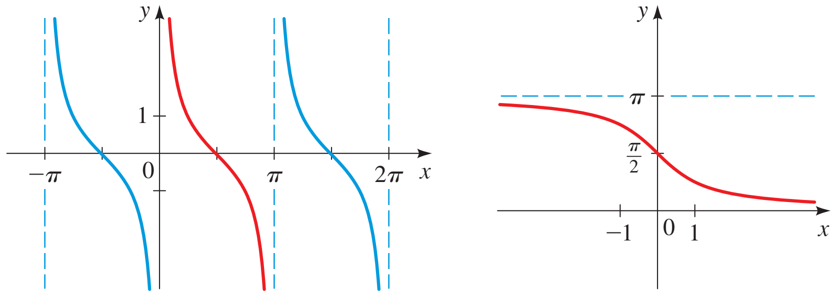
$y = \sec x,\ 0 \le x < \frac{\pi}{2},\ \pi \le x < \frac{3\pi}{2}$ $\quad y = \sec^{-1} x$

FIGURE 6 The inverse secant function



$y = \csc x,\ 0 < x \le \frac{\pi}{2},\ \pi < x \le \frac{3\pi}{2}$ $\quad y = \csc^{-1} x$

FIGURE 7 The inverse cosecant function



$y = \cot x,\ 0 < x < \pi$ $\quad y = \cot^{-1} x$

FIGURE 8 The inverse cotangent function

5.5 EXERCISES

CONCEPTS

1. **(a)** To define the inverse sine function, we restrict the domain of sine to the interval __________. On this interval the sine function is one-to-one, and its inverse function $\sin^{-1}$ is defined by $\sin^{-1} x = y \Leftrightarrow \sin$ _____ = _____. For example, $\sin^{-1}\frac{1}{2} =$ _____ because sin _____ = _____.

   (b) To define the inverse cosine function, we restrict the domain of cosine to the interval __________. On this interval the cosine function is one-to-one and its inverse function $\cos^{-1}$ is defined by $\cos^{-1} x = y \Leftrightarrow$ cos _____ = _____. For example, $\cos^{-1}\frac{1}{2} =$ _____ because cos _____ = _____.

2. The cancellation property $\sin^{-1}(\sin x) = x$ is valid for x in the interval __________. Which of the following is not true?

   (i) $\sin^{-1}\left(\sin \frac{\pi}{3}\right) = \frac{\pi}{3}$ (ii) $\sin^{-1}\left(\sin \frac{10\pi}{3}\right) = \frac{10\pi}{3}$

   (iii) $\sin^{-1}\left(\sin\left(-\frac{\pi}{4}\right)\right) = -\frac{\pi}{4}$

SKILLS

3–10 ■ Evaluating Inverse Trigonometric Functions Find the exact value of each expression, if it is defined.

3. **(a)** $\sin^{-1} 1$ **(b)** $\sin^{-1}\frac{\sqrt{3}}{2}$ **(c)** $\sin^{-1} 2$

4. (a) $\sin^{-1}(-1)$ (b) $\sin^{-1}\dfrac{\sqrt{2}}{2}$ (c) $\sin^{-1}(-2)$

5. (a) $\cos^{-1}(-1)$ (b) $\cos^{-1}\frac{1}{2}$ (c) $\cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

6. (a) $\cos^{-1}\dfrac{\sqrt{2}}{2}$ (b) $\cos^{-1} 1$ (c) $\cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$

7. (a) $\tan^{-1}(-1)$ (b) $\tan^{-1}\sqrt{3}$ (c) $\tan^{-1}\dfrac{\sqrt{3}}{3}$

8. (a) $\tan^{-1} 0$ (b) $\tan^{-1}(-\sqrt{3})$ (c) $\tan^{-1}\left(-\dfrac{\sqrt{3}}{3}\right)$

9. (a) $\cos^{-1}(-\frac{1}{2})$ (b) $\sin^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$ (c) $\tan^{-1} 1$

10. (a) $\cos^{-1} 0$ (b) $\sin^{-1} 0$ (c) $\sin^{-1}(-\frac{1}{2})$

11–22 ■ Inverse Trigonometric Functions with a Calculator Use a calculator to find an approximate value of each expression correct to five decimal places, if it is defined.

11. $\sin^{-1}\frac{2}{3}$
12. $\sin^{-1}(-\frac{8}{9})$
13. $\cos^{-1}(-\frac{3}{7})$
14. $\cos^{-1}(\frac{4}{9})$
15. $\cos^{-1}(-0.92761)$
16. $\sin^{-1}(0.13844)$
17. $\tan^{-1} 10$
18. $\tan^{-1}(-26)$
19. $\tan^{-1}(1.23456)$
20. $\cos^{-1}(1.23456)$
21. $\sin^{-1}(-0.25713)$
22. $\tan^{-1}(-0.25713)$

23–48 ■ Simplifying Expressions Involving Trigonometric Functions Find the exact value of the expression, if it is defined.

23. $\sin(\sin^{-1}\frac{1}{4})$
24. $\cos(\cos^{-1}\frac{2}{3})$
25. $\tan(\tan^{-1} 5)$
26. $\sin(\sin^{-1} 5)$
27. $\sin(\sin^{-1}\frac{3}{2})$
28. $\tan(\tan^{-1}\frac{3}{2})$
29. $\cos\left(\cos^{-1}\left(-\dfrac{1}{5}\right)\right)$
30. $\sin\left(\sin^{-1}\left(-\dfrac{3}{4}\right)\right)$
31. $\sin^{-1}\left(\sin\left(\dfrac{\pi}{4}\right)\right)$
32. $\cos^{-1}\left(\cos\left(\dfrac{\pi}{4}\right)\right)$
33. $\sin^{-1}\left(\sin\left(\dfrac{3\pi}{4}\right)\right)$
34. $\cos^{-1}\left(\cos\left(\dfrac{3\pi}{4}\right)\right)$
35. $\cos^{-1}\left(\cos\left(\dfrac{5\pi}{6}\right)\right)$
36. $\sin^{-1}\left(\sin\left(\dfrac{5\pi}{6}\right)\right)$
37. $\cos^{-1}\left(\cos\left(\dfrac{7\pi}{6}\right)\right)$
38. $\sin^{-1}\left(\sin\left(\dfrac{7\pi}{6}\right)\right)$
39. $\tan^{-1}\left(\tan\left(\dfrac{\pi}{4}\right)\right)$
40. $\tan^{-1}\left(\tan\left(-\dfrac{\pi}{3}\right)\right)$
41. $\tan^{-1}\left(\tan\left(\dfrac{2\pi}{3}\right)\right)$
42. $\sin^{-1}\left(\sin\left(\dfrac{11\pi}{4}\right)\right)$
43. $\tan(\sin^{-1}\frac{1}{2})$
44. $\cos(\sin^{-1} 0)$
45. $\cos\left(\sin^{-1}\dfrac{\sqrt{3}}{2}\right)$
46. $\tan\left(\sin^{-1}\dfrac{\sqrt{2}}{2}\right)$
47. $\sin(\tan^{-1}(-1))$
48. $\sin(\tan^{-1}(-\sqrt{3}))$

DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

49–50 ■ PROVE: Identities Involving Inverse Trigonometric Functions **(a)** Graph the function and make a conjecture, and **(b)** prove that your conjecture is true.

49. $y = \sin^{-1} x + \cos^{-1} x$
50. $y = \tan^{-1} x + \tan^{-1}\dfrac{1}{x}$

51. **DISCUSS: Two Different Compositions** Let f and g be the functions

$$f(x) = \sin(\sin^{-1} x)$$

and $$g(x) = \sin^{-1}(\sin x)$$

By the cancellation properties, $f(x) = x$ and $g(x) = x$ for suitable values of x. But these functions are not the same for all x. Graph both f and g to show how the functions differ. (Think carefully about the domain and range of $\sin^{-1}$).

5.6 MODELING HARMONIC MOTION

■ Simple Harmonic Motion ■ Damped Harmonic Motion ■ Phase and Phase Difference

Periodic behavior—behavior that repeats over and over again—is common in nature. Perhaps the most familiar example is the daily rising and setting of the sun, which results in the repetitive pattern of day, night, day, night, ... Another example is the daily variation of tide levels at the beach, which results in the repetitive pattern of high tide, low tide, high tide, low tide, ... Certain animal populations increase and decrease in a predictable periodic pattern: A large population exhausts the food supply, which causes the population to dwindle; this in turn results in a more plentiful food supply, which makes it possible for the population to increase; and the pattern then repeats over and over (see *Discovery Project: Predator/Prey Models* referenced on page 427).

Other common examples of periodic behavior involve motion that is caused by vibration or oscillation. A mass suspended from a spring that has been compressed and