4.1 EXPONENTIAL FUNCTIONS

■ Exponential Functions ■ Graphs of Exponential Functions ■ Compound Interest

In this chapter we study a new class of functions called *exponential functions*. For example,

$$f(x) = 2^x$$

is an exponential function (with base 2). Notice how quickly the values of this function increase.

$$f(3) = 2^3 = 8$$
$$f(10) = 2^{10} = 1024$$
$$f(30) = 2^{30} = 1{,}073{,}741{,}824$$

Compare this with the function $g(x) = x^2$, where $g(30) = 30^2 = 900$. The point is that when the variable is in the exponent, even a small change in the variable can cause a dramatic change in the value of the function.

■ Exponential Functions

To study exponential functions, we must first define what we mean by the exponential expression a^x when x is any real number. In Section 1.2 we defined a^x for $a > 0$ and x a rational number, but we have not yet defined irrational powers. So what is meant by $5^{\sqrt{3}}$ or 2^{π}? To define a^x when x is irrational, we approximate x by rational numbers.

For example, since

$$\sqrt{3} \approx 1.73205\ldots$$

is an irrational number, we successively approximate $a^{\sqrt{3}}$ by the following rational powers:

$$a^{1.7},\ a^{1.73},\ a^{1.732},\ a^{1.7320},\ a^{1.73205},\ \ldots$$

Intuitively, we can see that these rational powers of a are getting closer and closer to $a^{\sqrt{3}}$. It can be shown by using advanced mathematics that there is exactly one number that these powers approach. We define $a^{\sqrt{3}}$ to be this number.

For example, using a calculator, we find

$$5^{\sqrt{3}} \approx 5^{1.732}$$
$$\approx 16.2411\ldots$$

The more decimal places of $\sqrt{3}$ we use in our calculation, the better our approximation of $5^{\sqrt{3}}$.

The Laws of Exponents are listed on page 14.

It can be proved that the *Laws of Exponents are still true when the exponents are real numbers*.

EXPONENTIAL FUNCTIONS

The **exponential function with base a** is defined for all real numbers x by

$$f(x) = a^x$$

where $a > 0$ and $a \neq 1$.

We assume that $a \neq 1$ because the function $f(x) = 1^x = 1$ is just a constant function. Here are some examples of exponential functions:

$$\underbrace{f(x) = 2^x}_{\text{Base 2}} \qquad \underbrace{g(x) = 3^x}_{\text{Base 3}} \qquad \underbrace{h(x) = 10^x}_{\text{Base 10}}$$

EXAMPLE 1 ■ Evaluating Exponential Functions

Let $f(x) = 3^x$, and evaluate the following:

(a) $f(5)$ **(b)** $f(-\frac{2}{3})$

(c) $f(\pi)$ **(d)** $f(\sqrt{2})$

SOLUTION We use a calculator to obtain the values of f.

	Calculator keystrokes	Output
(a) $f(5) = 3^5 = 243$	3 ^ 5 ENTER	243
(b) $f(-\frac{2}{3}) = 3^{-2/3} \approx 0.4807$	3 ^ ((−) 2 ÷ 3) ENTER	0.4807498
(c) $f(\pi) = 3^{\pi} \approx 31.544$	3 ^ π ENTER	31.5442807
(d) $f(\sqrt{2}) = 3^{\sqrt{2}} \approx 4.7288$	3 ^ √ 2 ENTER	4.7288043

Now Try Exercise 7

■ Graphs of Exponential Functions

We first graph exponential functions by plotting points. We will see that the graphs of such functions have an easily recognizable shape.

EXAMPLE 2 ■ Graphing Exponential Functions by Plotting Points

Draw the graph of each function.

(a) $f(x) = 3^x$ **(b)** $g(x) = \left(\dfrac{1}{3}\right)^x$

SOLUTION We calculate values of $f(x)$ and $g(x)$ and plot points to sketch the graphs in Figure 1.

x	$f(x) = 3^x$	$g(x) = (\frac{1}{3})^x$
-3	$\frac{1}{27}$	27
-2	$\frac{1}{9}$	9
-1	$\frac{1}{3}$	3
0	1	1
1	3	$\frac{1}{3}$
2	9	$\frac{1}{9}$
3	27	$\frac{1}{27}$

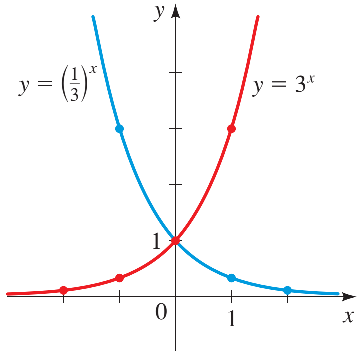


FIGURE 1

Notice that

$$g(x) = \left(\frac{1}{3}\right)^x = \frac{1}{3^x} = 3^{-x} = f(-x)$$

so we could have obtained the graph of g from the graph of f by reflecting in the y-axis.

Reflecting graphs is explained in Section 2.6.

Now Try Exercise 17

Figure 2 shows the graphs of the family of exponential functions $f(x) = a^x$ for various values of the base a. All of these graphs pass through the point $(0, 1)$ because

To see just how quickly $f(x) = 2^x$ increases, let's perform the following thought experiment. Suppose we start with a piece of paper that is a thousandth of an inch thick, and we fold it in half 50 times. Each time we fold the paper, the thickness of the paper stack doubles, so the thickness of the resulting stack would be $2^{50}/1000$ inches. How thick do you think that is? It works out to be more than 17 million miles!

$a^0 = 1$ for $a \neq 0$. You can see from Figure 2 that there are two kinds of exponential functions: If $0 < a < 1$, the exponential function decreases rapidly. If $a > 1$, the function increases rapidly (see the margin note).

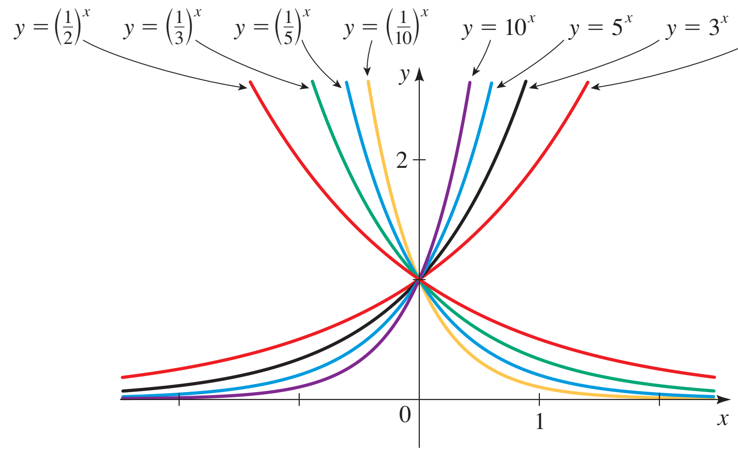


FIGURE 2 A family of exponential functions

See Section 3.6, page 295, where the arrow notation used here is explained.

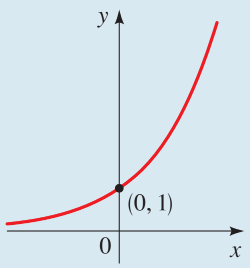
The x-axis is a horizontal asymptote for the exponential function $f(x) = a^x$. This is because when $a > 1$, we have $a^x \to 0$ as $x \to -\infty$, and when $0 < a < 1$, we have $a^x \to 0$ as $x \to \infty$ (see Figure 2). Also, $a^x > 0$ for all $x \in \mathbb{R}$, so the function $f(x) = a^x$ has domain $\mathbb{R}$ and range $(0, \infty)$. These observations are summarized in the following box.

GRAPHS OF EXPONENTIAL FUNCTIONS

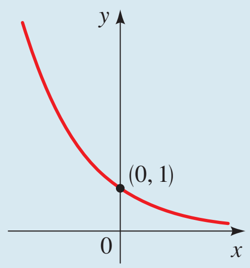
The exponential function

$$f(x) = a^x \qquad a > 0, a \neq 1$$

has domain $\mathbb{R}$ and range $(0, \infty)$. The line $y = 0$ (the x-axis) is a horizontal asymptote of f. The graph of f has one of the following shapes.



$f(x) = a^x$ for $a > 1$

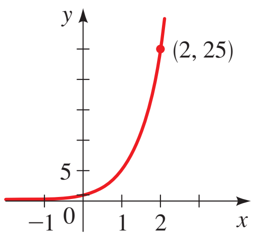


$f(x) = a^x$ for $0 < a < 1$

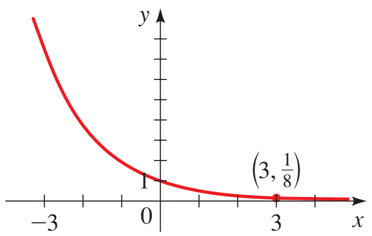
EXAMPLE 3 ■ Identifying Graphs of Exponential Functions

Find the exponential function $f(x) = a^x$ whose graph is given.

(a)



(b)



SOLUTION

(a) Since $f(2) = a^2 = 25$, we see that the base is $a = 5$. So $f(x) = 5^x$.

(b) Since $f(3) = a^3 = \frac{1}{8}$, we see that the base is $a = \frac{1}{2}$. So $f(x) = \left(\frac{1}{2}\right)^x$.

Now Try Exercise 21 ■

In the next example we see how to graph certain functions, not by plotting points, but by taking the basic graphs of the exponential functions in Figure 2 and applying the shifting and reflecting transformations of Section 2.6.

EXAMPLE 4 ■ Transformations of Exponential Functions

Use the graph of $f(x) = 2^x$ to sketch the graph of each function. State the domain, range, and asymptote.

(a) $g(x) = 1 + 2^x$ **(b)** $h(x) = -2^x$ **(c)** $k(x) = 2^{x-1}$

SOLUTION

Shifting and reflecting of graphs are explained in Section 2.6.

(a) To obtain the graph of $g(x) = 1 + 2^x$, we start with the graph of $f(x) = 2^x$ and shift it upward 1 unit to get the graph shown in Figure 3(a). From the graph we see that the domain of g is the set $\mathbb{R}$ of real numbers, the range is the interval $(1, \infty)$, and the line $y = 1$ is a horizontal asymptote.

(b) Again we start with the graph of $f(x) = 2^x$, but here we reflect in the x-axis to get the graph of $h(x) = -2^x$ shown in Figure 3(b). From the graph we see that the domain of h is the set $\mathbb{R}$ of all real numbers, the range is the interval $(-\infty, 0)$, and the line $y = 0$ is a horizontal asymptote.

(c) This time we start with the graph of $f(x) = 2^x$ and shift it to the right by 1 unit to get the graph of $k(x) = 2^{x-1}$ shown in Figure 3(c). From the graph we see that the domain of k is the set $\mathbb{R}$ of all real numbers, the range is the interval $(0, \infty)$, and the line $y = 0$ is a horizontal asymptote.

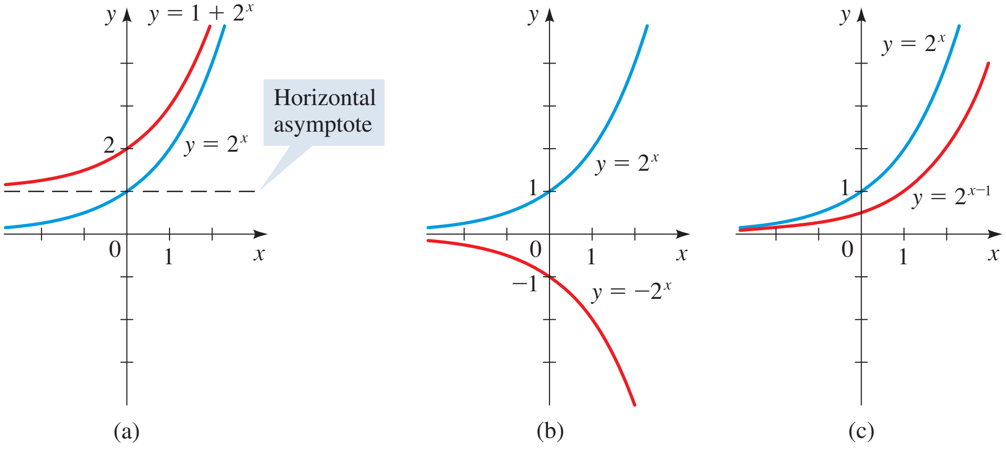


FIGURE 3

Now Try Exercises 27, 29, and 31 ■

EXAMPLE 5 ■ Comparing Exponential and Power Functions

Compare the rates of growth of the exponential function $f(x) = 2^x$ and the power function $g(x) = x^2$ by drawing the graphs of both functions in the following viewing rectangles.

(a) $[0, 3]$ by $[0, 8]$ **(b)** $[0, 6]$ by $[0, 25]$ **(c)** $[0, 20]$ by $[0, 1000]$

SOLUTION

(a) Figure 4(a) shows that the graph of $g(x) = x^2$ catches up with, and becomes higher than, the graph of $f(x) = 2^x$ at $x = 2$.

(b) The larger viewing rectangle in Figure 4(b) shows that the graph of $f(x) = 2^x$ overtakes that of $g(x) = x^2$ when $x = 4$.

(c) Figure 4(c) gives a more global view and shows that when x is large, $f(x) = 2^x$ is much larger than $g(x) = x^2$.

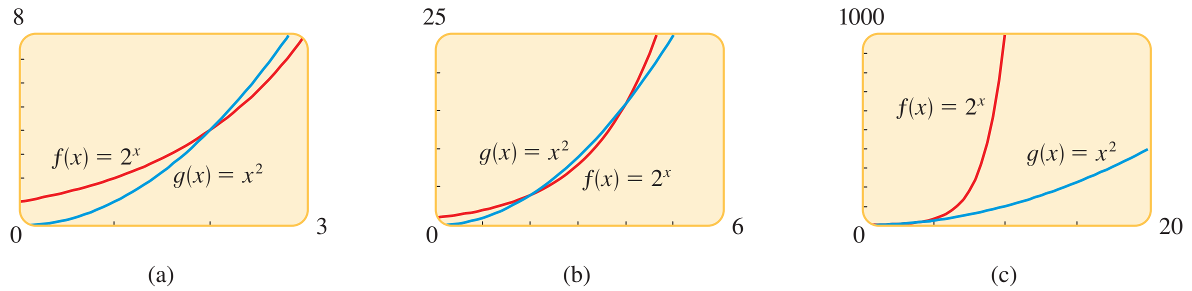


FIGURE 4

Now Try Exercise 45 ■

Compound Interest

Exponential functions occur in calculating compound interest. If an amount of money P, called the **principal**, is invested at an interest rate i per time period, then after one time period the interest is Pi, and the amount A of money is

$$A = P + Pi = P(1 + i)$$

If the interest is reinvested, then the new principal is $P(1 + i)$, and the amount after another time period is $A = P(1 + i)(1 + i) = P(1 + i)^2$. Similarly, after a third time period the amount is $A = P(1 + i)^3$. In general, after k periods the amount is

$$A = P(1 + i)^k$$

Notice that this is an exponential function with base $1 + i$.

If the annual interest rate is r and if interest is compounded n times per year, then in each time period the interest rate is $i = r/n$, and there are nt time periods in t years. This leads to the following formula for the amount after t years.

COMPOUND INTEREST

Compound interest is calculated by the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where $A(t)$ = amount after t years

P = principal

r = interest rate per year

n = number of times interest is compounded per year

t = number of years

r is often referred to as the *nominal annual interest rate*.

EXAMPLE 6 ■ Calculating Compound Interest

A sum of \$1000 is invested at an interest rate of 12% per year. Find the amounts in the account after 3 years if interest is compounded annually, semiannually, quarterly, monthly, and daily.

SOLUTION We use the compound interest formula with $P = \$1000$, $r = 0.12$, and $t = 3$.

Compounding	n	Amount after 3 years
Annual	1	$1000\left(1 + \frac{0.12}{1}\right)^{1(3)} = \$1404.93$
Semiannual	2	$1000\left(1 + \frac{0.12}{2}\right)^{2(3)} = \$1418.52$
Quarterly	4	$1000\left(1 + \frac{0.12}{4}\right)^{4(3)} = \$1425.76$
Monthly	12	$1000\left(1 + \frac{0.12}{12}\right)^{12(3)} = \$1430.77$
Daily	365	$1000\left(1 + \frac{0.12}{365}\right)^{365(3)} = \$1433.24$

Now Try Exercise 57 ■

If an investment earns compound interest, then the **annual percentage yield** (APY) is the *simple* interest rate that yields the same amount at the end of one year.

EXAMPLE 7 ■ Calculating the Annual Percentage Yield

Find the annual percentage yield for an investment that earns interest at a rate of 6% per year, compounded daily.

SOLUTION After one year, a principal P will grow to the amount

$$A = P\left(1 + \frac{0.06}{365}\right)^{365} = P(1.06183)$$

Simple interest is studied in Section 1.7.

The formula for simple interest is

$$A = P(1 + r)$$

Comparing, we see that $1 + r = 1.06183$, so $r = 0.06183$. Thus the annual percentage yield is 6.183%.

Now Try Exercise 63 ■

DISCOVERY PROJECT

So You Want to Be a Millionaire?

In this project we explore how rapidly the values of an exponential function increase by examining some real-world situations. For example, if you save a penny today, two pennies tomorrow, four pennies the next day, and so on, how long do you have to continue saving in this way before you become a millionaire? You can find out the surprising answer to this and other questions by completing this discovery project. You can find the project at **www.stewartmath.com**.

4.1 EXERCISES

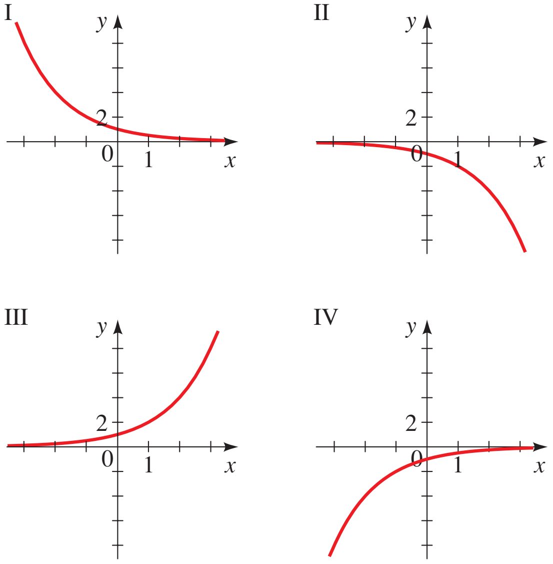
CONCEPTS

1. The function $f(x) = 5^x$ is an exponential function with base ________; $f(-2) =$ ________, $f(0) =$ ________, $f(2) =$ ________, and $f(6) =$ ________.

2. Match the exponential function with one of the graphs labeled I, II, III, or IV, shown below.

(a) $f(x) = 2^x$ **(b)** $f(x) = 2^{-x}$

(c) $f(x) = -2^x$ **(d)** $f(x) = -2^{-x}$



3. (a) To obtain the graph of $g(x) = 2^x - 1$, we start with the graph of $f(x) = 2^x$ and shift it ____________ (upward/downward) 1 unit.

(b) To obtain the graph of $h(x) = 2^{x-1}$, we start with the graph of $f(x) = 2^x$ and shift it to the ____________ (left/right) 1 unit.

4. In the formula $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ for compound interest the letters P, r, n, and t stand for ____________, ____________, ____________, and ____________, respectively, and $A(t)$ stands for ____________. So if \$100 is invested at an interest rate of 6% compounded quarterly, then the amount after 2 years is ____________.

5. The exponential function $f(x) = \left(\frac{1}{2}\right)^x$ has the ______________ asymptote $y =$ ________. This means that as $x \to \infty$, we have $\left(\frac{1}{2}\right)^x \to$ ________.

6. The exponential function $f(x) = \left(\frac{1}{2}\right)^x + 3$ has the ______________ asymptote $y =$ ________. This means that as $x \to \infty$, we have $\left(\frac{1}{2}\right)^x + 3 \to$ ________.

SKILLS

7–10 ■ Evaluating Exponential Functions Use a calculator to evaluate the function at the indicated values. Round your answers to three decimals.

7. $f(x) = 4^x$; $f\left(\frac{1}{2}\right), f(\sqrt{5}), f(-2), f(0.3)$

8. $f(x) = 3^{x-1}$; $f\left(\frac{1}{2}\right), f(2.5), f(-1), f\left(\frac{1}{4}\right)$

9. $g(x) = \left(\frac{1}{3}\right)^{x+1}$; $g\left(\frac{1}{2}\right), g(\sqrt{2}), g(-3.5), g(-1.4)$

10. $g(x) = \left(\frac{4}{3}\right)^{3x}$; $g\left(-\frac{1}{2}\right), g(\sqrt{6}), g(-3), g\left(\frac{4}{3}\right)$

11–16 ■ Graphing Exponential Functions Sketch the graph of the function by making a table of values. Use a calculator if necessary.

11. $f(x) = 2^x$ **12.** $g(x) = 8^x$

13. $f(x) = \left(\frac{1}{3}\right)^x$ **14.** $h(x) = (1.1)^x$

15. $g(x) = 3(1.3)^x$ **16.** $h(x) = 2\left(\frac{1}{4}\right)^x$

17–20 ■ Graphing Exponential Functions Graph both functions on one set of axes.

17. $f(x) = 2^x$ and $g(x) = 2^{-x}$

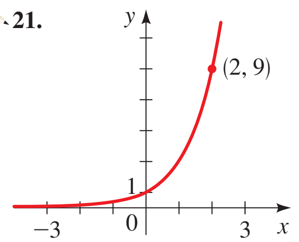
18. $f(x) = 3^{-x}$ and $g(x) = \left(\frac{1}{3}\right)^x$

19. $f(x) = 4^x$ and $g(x) = 7^x$

20. $f(x) = \left(\frac{3}{4}\right)^x$ and $g(x) = 1.5^x$

21–24 ■ Exponential Functions from a Graph Find the exponential function $f(x) = a^x$ whose graph is given.

21.



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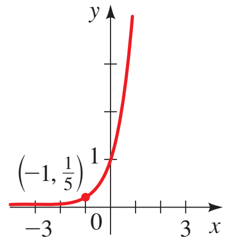

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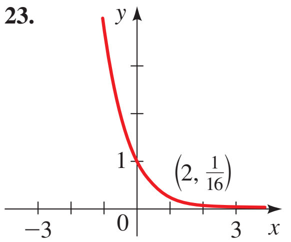

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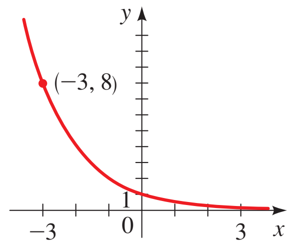
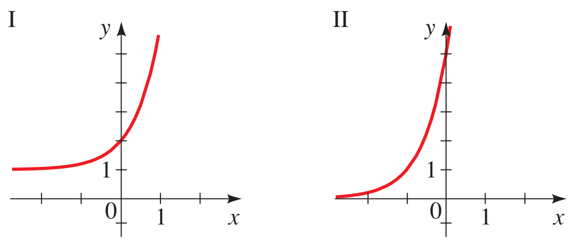


25–26 ■ Exponential Functions from a Graph Match the exponential function with one of the graphs labeled I or II.

25. $f(x) = 5^{x+1}$ **26.** $f(x) = 5^x + 1$



27–40 ■ Graphing Exponential Functions Graph the function, not by plotting points, but by starting from the graphs in Figure 2. State the domain, range, and asymptote.

27. $g(x) = 2^x - 3$ **28.** $h(x) = 4 + \left(\frac{1}{2}\right)^x$

29. $f(x) = -3^x$ **30.** $f(x) = 10^{-x}$

31. $f(x) = 10^{x+3}$ **32.** $g(x) = 2^{x-3}$

33. $y = 5^{-x} + 1$ **34.** $h(x) = 6 - 3^x$

35. $y = 2 - \left(\frac{1}{3}\right)^x$ **36.** $y = 5^{-x} - 3$

37. $h(x) = 2^{x-4} + 1$ **38.** $y = 3 - 10^{x-1}$

39. $g(x) = 1 - 3^{-x}$ **40.** $y = 3 - \left(\frac{1}{5}\right)^x$

41–42 ■ Comparing Exponential Functions In these exercises we compare the graphs of two exponential functions.

41. **(a)** Sketch the graphs of $f(x) = 2^x$ and $g(x) = 3(2^x)$.
(b) How are the graphs related?

42. **(a)** Sketch the graphs of $f(x) = 9^{x/2}$ and $g(x) = 3^x$.
(b) Use the Laws of Exponents to explain the relationship between these graphs.

43–44 ■ Comparing Exponential and Power Functions Compare the graphs of the power function f and exponential function g by evaluating both of them for $x = 0, 1, 2, 3, 4, 6, 8,$ and 10. Then draw the graphs of f and g on the same set of axes.

43. $f(x) = x^3$; $g(x) = 3^x$ **44.** $f(x) = x^4$; $g(x) = 4^x$

45–46 ■ Comparing Exponential and Power Functions In these exercises we use a graphing calculator to compare the rates of growth of the graphs of a power function and an exponential function.

45. **(a)** Compare the rates of growth of the functions $f(x) = 2^x$ and $g(x) = x^5$ by drawing the graphs of both functions in the following viewing rectangles.
(i) $[0, 5]$ by $[0, 20]$
(ii) $[0, 25]$ by $[0, 10^7]$
(iii) $[0, 50]$ by $[0, 10^8]$
(b) Find the solutions of the equation $2^x = x^5$, rounded to one decimal place.

46. **(a)** Compare the rates of growth of the functions $f(x) = 3^x$ and $g(x) = x^4$ by drawing the graphs of both functions in the following viewing rectangles:
(i) $[-4, 4]$ by $[0, 20]$
(ii) $[0, 10]$ by $[0, 5000]$
(iii) $[0, 20]$ by $[0, 10^5]$
(b) Find the solutions of the equation $3^x = x^4$, rounded to two decimal places.

SKILLS Plus

47–48 ■ Families of Functions Draw graphs of the given family of functions for $c = 0.25, 0.5, 1, 2, 4$. How are the graphs related?

47. $f(x) = c2^x$ **48.** $f(x) = 2^{cx}$

49–50 ■ Getting Information from a Graph Find, rounded to two decimal places, **(a)** the intervals on which the function is increasing or decreasing and **(b)** the range of the function.

49. $y = 10^{x-x^2}$ **50.** $y = x2^x$

51–52 ■ Difference Quotients These exercises involve a difference quotient for an exponential function.

51. If $f(x) = 10^x$, show that

$$\frac{f(x+h) - f(x)}{h} = 10^x\left(\frac{10^h - 1}{h}\right)$$

52. If $f(x) = 3^{x-1}$, show that

$$\frac{f(x+h) - f(x)}{h} = 3^{x-1}\left(\frac{3^h - 1}{h}\right)$$

APPLICATIONS

53. Bacteria Growth A bacteria culture contains 1500 bacteria initially and doubles every hour.
(a) Find a function N that models the number of bacteria after t hours.
(b) Find the number of bacteria after 24 hours.

54. Mouse Population A certain breed of mouse was introduced onto a small island with an initial population of 320 mice, and scientists estimate that the mouse population is doubling every year.
(a) Find a function N that models the number of mice after t years.
(b) Estimate the mouse population after 8 years.

55–56 ■ Compound Interest An investment of \$5000 is deposited into an account in which interest is compounded monthly. Complete the table by filling in the amounts to which the investment grows at the indicated times or interest rates.

55. $r = 4\%$

Time (years)	Amount
1	
2	
3	
4	
5	
6	

56. $t = 5$ years

Rate per year	Amount
1%	
2%	
3%	
4%	
5%	
6%	

57. Compound Interest If \$10,000 is invested at an interest rate of 3% per year, compounded semiannually, find the value of the investment after the given number of years.
(a) 5 years **(b)** 10 years **(c)** 15 years

58. Compound Interest If \$2500 is invested at an interest rate of 2.5% per year, compounded daily, find the value of the investment after the given number of years.
(a) 2 years **(b)** 3 years **(c)** 6 years

59. Compound Interest If \$500 is invested at an interest rate of 3.75% per year, compounded quarterly, find the value of the investment after the given number of years.
(a) 1 year **(b)** 2 years **(c)** 10 years

60. Compound Interest If \$4000 is borrowed at a rate of 5.75% interest per year, compounded quarterly, find the amount due at the end of the given number of years.
(a) 4 years **(b)** 6 years **(c)** 8 years