

3.6 RATIONAL FUNCTIONS

■ Rational Functions and Asymptotes ■ Transformations of $y = 1/x$ ■ Asymptotes of Rational Functions ■ Graphing Rational Functions ■ Common Factors in Numerator and Denominator ■ Slant Asymptotes and End Behavior ■ Applications

A rational function is a function of the form

$$r(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. We assume that $P(x)$ and $Q(x)$ have no factor in common. Even though rational functions are constructed from polynomials, their graphs look quite different from the graphs of polynomial functions.

■ Rational Functions and Asymptotes

Domains of rational expressions are discussed in Section 1.4.

The *domain* of a rational function consists of all real numbers x except those for which the denominator is zero. When graphing a rational function, we must pay special attention to the behavior of the graph near those x -values. We begin by graphing a very simple rational function.

EXAMPLE 1 ■ A Simple Rational Function

Graph the rational function $f(x) = 1/x$, and state the domain and range.

SOLUTION The function f is not defined for $x = 0$. The following tables show that when x is close to zero, the value of $|f(x)|$ is large, and the closer x gets to zero, the larger $|f(x)|$ gets.

For positive real numbers,

$$\frac{1}{\text{BIG NUMBER}} = \text{small number}$$

$$\frac{1}{\text{small number}} = \text{BIG NUMBER}$$

x	$f(x)$
-0.1	-10
-0.01	-100
-0.00001	-100,000

Approaching 0^-

Approaching $-\infty$

x	$f(x)$
0.1	10
0.01	100
0.00001	100,000

Approaching 0^+

Approaching ∞

We describe this behavior in words and in symbols as follows. The first table shows that as x approaches 0 from the left, the values of $y = f(x)$ decrease without bound. In symbols,

$$f(x) \rightarrow -\infty \quad \text{as} \quad x \rightarrow 0^- \quad \text{“}y \text{ approaches negative infinity as } x \text{ approaches 0 from the left”}$$



© silver-john/Shutterstock.com

DISCOVERY PROJECT

Managing Traffic

A highway engineer wants to determine the optimal safe driving speed for a road. The higher the speed limit, the more cars the road can accommodate, but safety requires a greater following distance at higher speeds. In this project we find a rational function that models the carrying capacity of a road at a given traffic speed. The model can be used to determine the speed limit at which the road has its maximum carrying capacity. You can find the project at www.stewartmath.com.

The second table shows that as x approaches 0 from the right, the values of $f(x)$ increase without bound. In symbols,

$$f(x) \rightarrow \infty \text{ as } x \rightarrow 0^+$$

“y approaches infinity as x approaches 0 from the right”

The next two tables show how $f(x)$ changes as $|x|$ becomes large.

x	$f(x)$	x	$f(x)$
-10	-0.1	10	0.1
-100	-0.01	100	0.01
-100,000	-0.00001	100,000	0.00001

Approaching $-\infty$

Approaching 0

Approaching ∞

Approaching 0

These tables show that as $|x|$ becomes large, the value of $f(x)$ gets closer and closer to zero. We describe this situation in symbols by writing

$$f(x) \rightarrow 0 \text{ as } x \rightarrow -\infty$$

and

$$f(x) \rightarrow 0 \text{ as } x \rightarrow \infty$$

Using the information in these tables and plotting a few additional points, we obtain the graph shown in Figure 1.

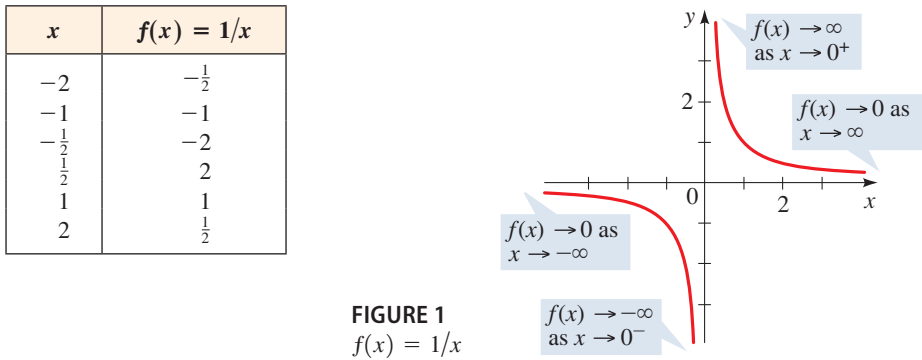


FIGURE 1
 $f(x) = 1/x$

Obtaining the domain and range of a function from its graph is explained in Section 2.3, page 171.

The function f is defined for all values of x other than 0, so the domain is $\{x \mid x \neq 0\}$. From the graph we see that the range is $\{y \mid y \neq 0\}$.

 Now Try Exercise 9

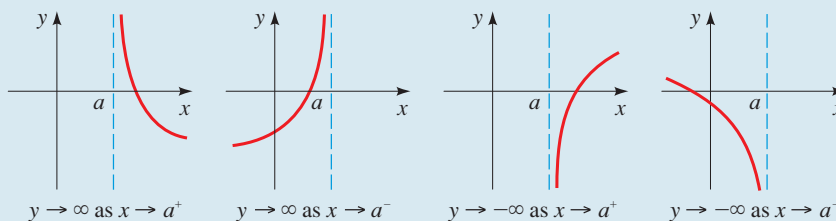
In Example 1 we used the following **arrow notation**.

Symbol	Meaning
$x \rightarrow a^-$	x approaches a from the left
$x \rightarrow a^+$	x approaches a from the right
$x \rightarrow -\infty$	x goes to negative infinity; that is, x decreases without bound
$x \rightarrow \infty$	x goes to infinity; that is, x increases without bound

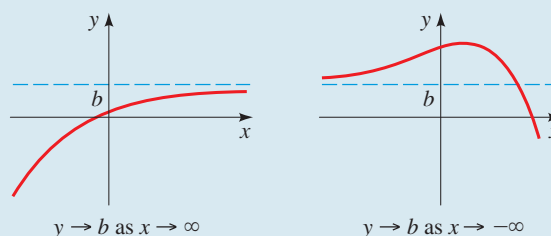
The line $x = 0$ is called a *vertical asymptote* of the graph in Figure 1, and the line $y = 0$ is a *horizontal asymptote*. Informally speaking, an asymptote of a function is a line to which the graph of the function gets closer and closer as one travels along that line.

DEFINITION OF VERTICAL AND HORIZONTAL ASYMPTOTES

1. The line $x = a$ is a **vertical asymptote** of the function $y = f(x)$ if y approaches $\pm\infty$ as x approaches a from the right or left.



2. The line $y = b$ is a **horizontal asymptote** of the function $y = f(x)$ if y approaches b as x approaches $\pm\infty$.



Recall that for a rational function $R(x) = P(x)/Q(x)$, we assume that $P(x)$ and $Q(x)$ have no factor in common.

A rational function has vertical asymptotes where the function is undefined, that is, where the denominator is zero.

■ Transformations of $y = 1/x$

A rational function of the form

$$r(x) = \frac{ax + b}{cx + d}$$

can be graphed by shifting, stretching, and/or reflecting the graph of $f(x) = 1/x$ shown in Figure 1, using the transformations studied in Section 2.6. (Such functions are called *linear fractional transformations*.)

EXAMPLE 2 ■ Using Transformations to Graph Rational Functions

Graph each rational function, and state the domain and range.

(a) $r(x) = \frac{2}{x-3}$ (b) $s(x) = \frac{3x+5}{x+2}$

SOLUTION

(a) Let $f(x) = 1/x$. Then we can express r in terms of f as follows:

$$\begin{aligned} r(x) &= \frac{2}{x-3} \\ &= 2\left(\frac{1}{x-3}\right) && \text{Factor 2} \\ &= 2(f(x-3)) && \text{Since } f(x) = 1/x \end{aligned}$$

From this form we see that the graph of r is obtained from the graph of f by shifting 3 units to the right and stretching vertically by a factor of 2. Thus r has vertical asymptote $x = 3$ and horizontal asymptote $y = 0$. The graph of r is shown in Figure 2.

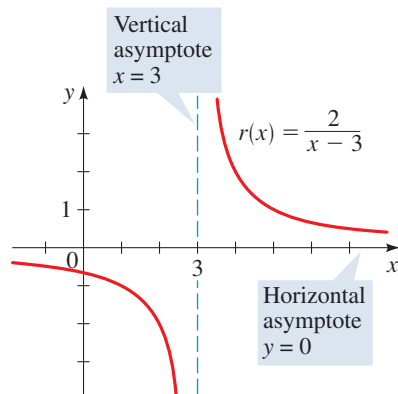


FIGURE 2

$$\begin{array}{r} 3 \\ x+2 \overline{) 3x+5} \\ \underline{3x+6} \\ -1 \end{array}$$

The function r is defined for all x other than 3, so the domain is $\{x \mid x \neq 3\}$. From the graph we see that the range is $\{y \mid y \neq 0\}$.

- (b) Using long division (see the margin), we get $s(x) = 3 - \frac{1}{x+2}$. Thus we can express s in terms of f as follows.

$$\begin{aligned} s(x) &= 3 - \frac{1}{x+2} \\ &= -\frac{1}{x+2} + 3 && \text{Rearrange terms} \\ &= -f(x+2) + 3 && \text{Since } f(x) = 1/x \end{aligned}$$

From this form we see that the graph of s is obtained from the graph of f by shifting 2 units to the left, reflecting in the x -axis, and shifting upward 3 units. Thus s has vertical asymptote $x = -2$ and horizontal asymptote $y = 3$. The graph of s is shown in Figure 3.

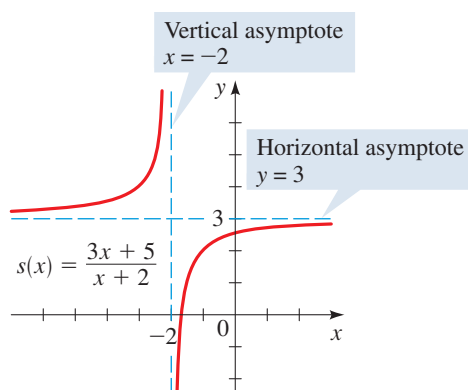


FIGURE 3

The function s is defined for all x other than -2 , so the domain is $\{x \mid x \neq -2\}$. From the graph we see that the range is $\{y \mid y \neq 3\}$.

 **Now Try Exercises 15 and 17**

■ Asymptotes of Rational Functions

The methods of Example 2 work only for simple rational functions. To graph more complicated ones, we need to take a closer look at the behavior of a rational function near its vertical and horizontal asymptotes.

EXAMPLE 3 ■ Asymptotes of a Rational Function

Graph $r(x) = \frac{2x^2 - 4x + 5}{x^2 - 2x + 1}$, and state the domain and range.

SOLUTION

Vertical asymptote. We first factor the denominator

$$r(x) = \frac{2x^2 - 4x + 5}{(x-1)^2}$$

The line $x = 1$ is a vertical asymptote because the denominator of r is zero when $x = 1$.

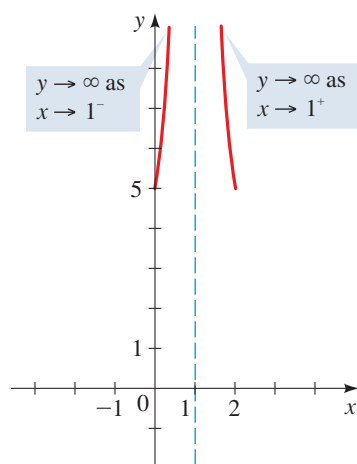


FIGURE 4

To see what the graph of r looks like near the vertical asymptote, we make tables of values for x -values to the left and to the right of 1. From the tables shown below we see that

$$y \rightarrow \infty \text{ as } x \rightarrow 1^- \quad \text{and} \quad y \rightarrow \infty \text{ as } x \rightarrow 1^+$$

 $x \rightarrow 1^-$

x	y
0	5
0.5	14
0.9	302
0.99	30,002

Approaching 1^- Approaching ∞
 $x \rightarrow 1^+$

x	y
2	5
1.5	14
1.1	302
1.01	30,002

Approaching 1^+ Approaching ∞

Thus near the vertical asymptote $x = 1$, the graph of r has the shape shown in Figure 4.

Horizontal asymptote. The horizontal asymptote is the value that y approaches as $x \rightarrow \pm\infty$. To help us find this value, we divide both numerator and denominator by x^2 , the highest power of x that appears in the expression:

$$y = \frac{2x^2 - 4x + 5}{x^2 - 2x + 1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \frac{2 - \frac{4}{x} + \frac{5}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}}$$

The fractional expressions $\frac{4}{x}$, $\frac{5}{x^2}$, $\frac{2}{x}$, and $\frac{1}{x^2}$ all approach 0 as $x \rightarrow \pm\infty$ (see Exercise 90, Section 1.1, page 12). So as $x \rightarrow \pm\infty$, we have

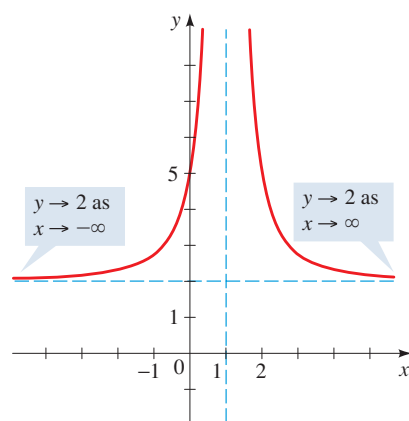


FIGURE 5

$$r(x) = \frac{2x^2 - 4x + 5}{x^2 - 2x + 1}$$

These terms approach 0

$$y = \frac{2 - \frac{4}{x} + \frac{5}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}} \longrightarrow \frac{2 - 0 + 0}{1 - 0 + 0} = 2$$

These terms approach 0

Thus the horizontal asymptote is the line $y = 2$.

Since the graph must approach the horizontal asymptote, we can complete it as in Figure 5.

Domain and range. The function r is defined for all values of x other than 1, so the domain is $\{x \mid x \neq 1\}$. From the graph we see that the range is $\{y \mid y > 2\}$.

Now Try Exercise 45

From Example 3 we see that the horizontal asymptote is determined by the leading coefficients of the numerator and denominator, since after dividing through by x^2 (the highest power of x), all other terms approach zero. In general, if $r(x) = P(x)/Q(x)$ and

the degrees of P and Q are the same (both n , say), then dividing both numerator and denominator by x^n shows that the horizontal asymptote is

$$y = \frac{\text{leading coefficient of } P}{\text{leading coefficient of } Q}$$

The following box summarizes the procedure for finding asymptotes.

FINDING ASYMPTOTES OF RATIONAL FUNCTIONS

Let r be the rational function

$$r(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}$$

1. The vertical asymptotes of r are the lines $x = a$, where a is a zero of the denominator.
2. (a) If $n < m$, then r has horizontal asymptote $y = 0$.
 (b) If $n = m$, then r has horizontal asymptote $y = \frac{a_n}{b_m}$.
 (c) If $n > m$, then r has no horizontal asymptote.

Recall that for a rational function $R(x) = P(x)/Q(x)$ we assume that $P(x)$ and $Q(x)$ have no factor in common. (See page 295.)

EXAMPLE 4 ■ Asymptotes of a Rational Function

Find the vertical and horizontal asymptotes of $r(x) = \frac{3x^2 - 2x - 1}{2x^2 + 3x - 2}$.

SOLUTION

Vertical asymptotes. We first factor

$$r(x) = \frac{3x^2 - 2x - 1}{(2x - 1)(x + 2)}$$

This factor is 0
when $x = \frac{1}{2}$

This factor is 0
when $x = -2$

The vertical asymptotes are the lines $x = \frac{1}{2}$ and $x = -2$.

Horizontal asymptote. The degrees of the numerator and denominator are the same, and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{3}{2}$$

Thus the horizontal asymptote is the line $y = \frac{3}{2}$.

To confirm our results, we graph r using a graphing calculator (see Figure 6).

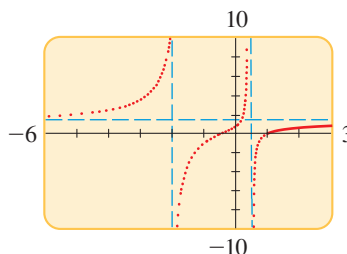


FIGURE 6

$$r(x) = \frac{3x^2 - 2x - 1}{2x^2 + 3x - 2}$$

Graph is drawn using dot mode to avoid extraneous lines.

 Now Try Exercises 33 and 35

■ Graphing Rational Functions

We have seen that asymptotes are important when graphing rational functions. In general, we use the following guidelines to graph rational functions.

A fraction is 0 only if its numerator is 0.

SKETCHING GRAPHS OF RATIONAL FUNCTIONS

- Factor.** Factor the numerator and denominator.
- Intercepts.** Find the x -intercepts by determining the zeros of the numerator and the y -intercept from the value of the function at $x = 0$.
- Vertical Asymptotes.** Find the vertical asymptotes by determining the zeros of the denominator, and then see whether $y \rightarrow \infty$ or $y \rightarrow -\infty$ on each side of each vertical asymptote by using test values.
- Horizontal Asymptote.** Find the horizontal asymptote (if any), using the procedure described in the box on page 300.
- Sketch the Graph.** Graph the information provided by the first four steps. Then plot as many additional points as needed to fill in the rest of the graph of the function.

EXAMPLE 5 ■ Graphing a Rational Function

Graph $r(x) = \frac{2x^2 + 7x - 4}{x^2 + x - 2}$, and state the domain and range.

SOLUTION We factor the numerator and denominator, find the intercepts and asymptotes, and sketch the graph.

Factor. $y = \frac{(2x - 1)(x + 4)}{(x - 1)(x + 2)}$

x -Intercepts. The x -intercepts are the zeros of the numerator, $x = \frac{1}{2}$ and $x = -4$.

y -Intercept. To find the y -intercept, we substitute $x = 0$ into the original form of the function.

$$r(0) = \frac{2(0)^2 + 7(0) - 4}{(0)2 + (0) - 2} = \frac{-4}{-2} = 2$$

The y -intercept is 2.

Vertical asymptotes. The vertical asymptotes occur where the denominator is 0, that is, where the function is undefined. From the factored form we see that the vertical asymptotes are the lines $x = 1$ and $x = -2$.

When choosing test values, we must make sure that there is no x -intercept between the test point and the vertical asymptote.

Behavior near vertical asymptotes. We need to know whether $y \rightarrow \infty$ or $y \rightarrow -\infty$ on each side of each vertical asymptote. To determine the sign of y for x -values near the vertical asymptotes, we use test values. For instance, as $x \rightarrow 1^-$, we use a test value close to and to the left of 1 ($x = 0.9$, say) to check whether y is positive or negative to the left of $x = 1$.

$$y = \frac{(2(0.9) - 1)((0.9) + 4)}{((0.9) - 1)((0.9) + 2)} \quad \text{whose sign is} \quad \frac{(+)(+)}{(-)(+)} \quad (\text{negative})$$

So $y \rightarrow -\infty$ as $x \rightarrow 1^-$. On the other hand, as $x \rightarrow 1^+$, we use a test value close to and to the right of 1 ($x = 1.1$, say), to get

$$y = \frac{(2(1.1) - 1)((1.1) + 4)}{((1.1) - 1)((1.1) + 2)} \quad \text{whose sign is} \quad \frac{(+)(+)}{(+)(+)} \quad (\text{positive})$$

Mathematics in the Modern World

Unbreakable Codes

If you read spy novels, you know about secret codes and how the hero “breaks” the code. Today secret codes have a much more common use. Most of the information that is stored on computers is coded to prevent unauthorized use. For example, your banking records, medical records, and school records are coded. Many cellular and cordless phones code the signal carrying your voice so that no one can listen in. Fortunately, because of recent advances in mathematics, today’s codes are “unbreakable.”

Modern codes are based on a simple principle: Factoring is much harder than multiplying. For example, try multiplying 78 and 93; now try factoring 9991. It takes a long time to factor 9991 because it is a product of two primes 97×103 , so to factor it, we have to find one of these primes. Now imagine trying to factor a number N that is the product of two primes p and q , each about 200 digits long. Even the fastest computers would take many millions of years to factor such a number! But the same computer would take less than a second to multiply two such numbers. This fact was used by Ron Rivest, Adi Shamir, and Leonard Adleman in the 1970s to devise the RSA code. Their code uses an extremely large number to encode a message but requires us to know its factors to decode it. As you can see, such a code is practically unbreakable.

The RSA code is an example of a “public key encryption” code. In such codes, anyone can code a message using a publicly known procedure based on N , but to decode the message, they must know p and q , the factors of N . When the RSA code was developed, it was thought that a carefully selected 80-digit number would provide an unbreakable code. But interestingly, recent advances in the study of factoring have made much larger numbers necessary.

So $y \rightarrow \infty$ as $x \rightarrow 1^+$. The other entries in the following table are calculated similarly.

As $x \rightarrow$	-2^-	-2^+	1^-	1^+
the sign of $y = \frac{(2x - 1)(x + 4)}{(x - 1)(x + 2)}$ is	$\frac{(-)(+)}{(-)(-)}$	$\frac{(-)(+)}{(-)(+)}$	$\frac{(+)(+)}{(-)(+)}$	$\frac{(+)(+)}{(+)(+)}$
so $y \rightarrow$	$-\infty$	∞	$-\infty$	∞

Horizontal asymptote. The degrees of the numerator and denominator are the same, and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{2}{1} = 2$$

Thus the horizontal asymptote is the line $y = 2$.

Graph. We use the information we have found, together with some additional values, to sketch the graph in Figure 7.

x	y
-6	0.93
-3	-1.75
-1	4.50
1.5	6.29
2	4.50
3	3.50

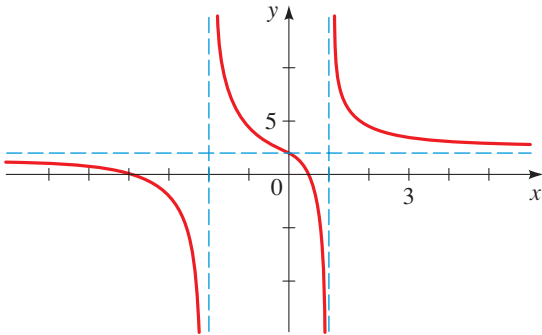


FIGURE 7
 $r(x) = \frac{2x^2 + 7x - 4}{x^2 + x - 2}$

Domain and range. The domain is $\{x \mid x \neq 1, x \neq -2\}$. From the graph we see that the range is all real numbers.

Now Try Exercise 53

EXAMPLE 6 ■ Graphing a Rational Function

Graph the rational function $r(x) = \frac{x^2 - 4}{2x^2 + 2x}$, and state the domain and range.

SOLUTION

Factor. $y = \frac{(x + 2)(x - 2)}{2x(x + 1)}$

x-intercepts. -2 and 2 , from $x + 2 = 0$ and $x - 2 = 0$

y-intercept. None, because $r(0)$ is undefined

Vertical asymptotes. $x = 0$ and $x = -1$, from the zeros of the denominator

Behavior near vertical asymptote.

As $x \rightarrow$	-1^-	-1^+	0^-	0^+
the sign of $y = \frac{(x+2)(x-2)}{2x(x+1)}$ is	$\frac{(+)(-)}{(-)(-)}$	$\frac{(+)(-)}{(-)(+)}$	$\frac{(+)(-)}{(-)(+)}$	$\frac{(+)(-)}{(+)(+)}$
so $y \rightarrow$	$-\infty$	∞	∞	$-\infty$

Horizontal asymptote. $y = \frac{1}{2}$, because the degree of the numerator and the degree of the denominator are the same and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{1}{2}$$

Graph. We use the information we have found, together with some additional values, to sketch the graph in Figure 8.

x	y
-0.9	17.72
-0.5	7.50
-0.45	7.67
-0.4	8.00
-0.3	9.31
-0.1	22.17

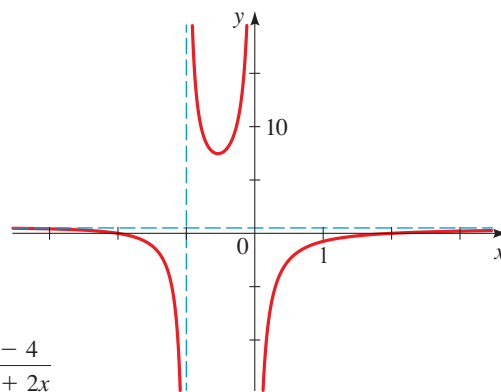


FIGURE 8

$$r(x) = \frac{x^2 - 4}{2x^2 + 2x}$$

Domain and range. The domain is $\{x \mid x \neq 0, x \neq -1\}$. From the graph we see that the range is $\{x \mid x < \frac{1}{2} \text{ or } x > 7.5\}$.



Now Try Exercise 55

EXAMPLE 7 ■ Graphing a Rational Function

Graph $r(x) = \frac{5x + 21}{x^2 + 10x + 25}$, and state the domain and range.

SOLUTION

Factor. $y = \frac{5x + 21}{(x + 5)^2}$

x-Intercept. $-\frac{21}{5}$, from $5x + 21 = 0$

y-Intercept. $\frac{21}{25}$, because $r(0) = \frac{5 \cdot 0 + 21}{0^2 + 10 \cdot 0 + 25}$

$$= \frac{21}{25}$$

Vertical asymptote. $x = -5$, from the zeros of the denominator

Behavior near vertical asymptote.

As $x \rightarrow$	-5^-	-5^+
the sign of $y = \frac{5x + 21}{(x + 5)^2}$ is	$\frac{(-)}{(-)(-)}$	$\frac{(-)}{(+)(+)}$
so $y \rightarrow$	$-\infty$	$-\infty$

Horizontal asymptote. $y = 0$, because the degree of the numerator is less than the degree of the denominator

Graph. We use the information we have found, together with some additional values, to sketch the graph in Figure 9.

x	y
-15	-0.5
-10	-1.2
-3	1.5
-1	1.0
3	0.6
5	0.5
10	0.3

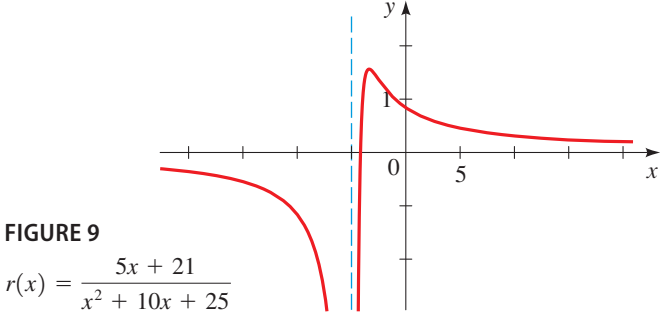


FIGURE 9

$$r(x) = \frac{5x + 21}{x^2 + 10x + 25}$$

Domain and range. The domain is $\{x \mid x \neq -5\}$. From the graph we see that the range is approximately the interval $(-\infty, 1.6]$.

 **Now Try Exercise 59**

 From the graph in Figure 9 we see that, **contrary to common misconception, a graph may cross a horizontal asymptote**. The graph in Figure 9 crosses the x -axis (the horizontal asymptote) from below, reaches a maximum value near $x = -3$, and then approaches the x -axis from above as $x \rightarrow \infty$.

Common Factors in Numerator and Denominator

We have adopted the convention that the numerator and denominator of a rational function have no factor in common. If $s(x) = p(x)/q(x)$ and if p and q do have a factor in common, then we may cancel that factor, but only for those values of x for which that factor is *not zero* (because division by zero is not defined). Since s is not defined at those values of x , its graph has a “hole” at those points, as the following example illustrates.

EXAMPLE 8 ■ Common Factor in Numerator and Denominator

Graph the following functions.

(a) $s(x) = \frac{x - 3}{x^2 - 3x}$ (b) $t(x) = \frac{x^3 - 2x^2}{x - 2}$

SOLUTION

(a) We factor the numerator and denominator:

$$s(x) = \frac{x - 3}{x^2 - 3x} = \frac{\cancel{(x - 3)}}{x\cancel{(x - 3)}} = \frac{1}{x} \quad \text{for } x \neq 3$$

So s has the same graph as the rational function $r(x) = 1/x$ but with a “hole” when x is 3, as shown in Figure 10(a).

(b) We factor the numerator and denominator:

$$t(x) = \frac{x^3 - 2x^2}{x - 2} = \frac{x^2(\cancel{x-2})}{\cancel{x-2}} = x^2 \quad \text{for } x \neq 2$$

So the graph of t is the same as the graph of $r(x) = x^2$ but with a “hole” when x is 2, as shown in Figure 10(b).

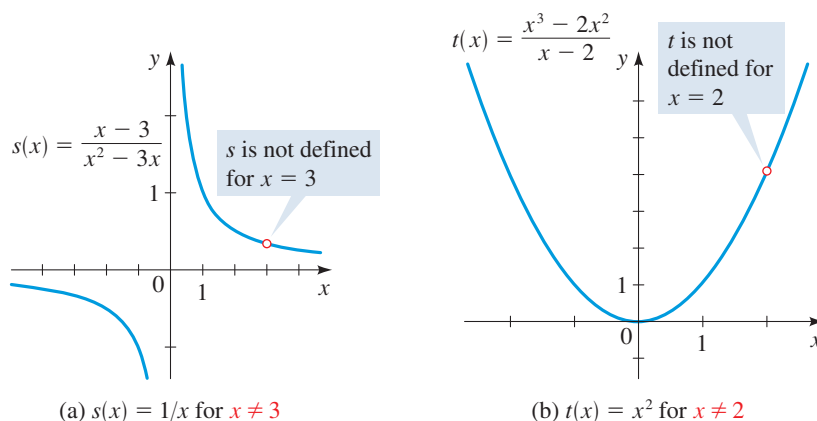


FIGURE 10 Graphs with “holes”

Now Try Exercise 63

■ Slant Asymptotes and End Behavior

If $r(x) = P(x)/Q(x)$ is a rational function in which the degree of the numerator is one more than the degree of the denominator, we can use the Division Algorithm to express the function in the form

$$r(x) = ax + b + \frac{R(x)}{Q(x)}$$

where the degree of R is less than the degree of Q and $a \neq 0$. This means that as $x \rightarrow \pm\infty$, $R(x)/Q(x) \rightarrow 0$, so for large values of $|x|$ the graph of $y = r(x)$ approaches the graph of the line $y = ax + b$. In this situation we say that $y = ax + b$ is a **slant asymptote**, or an **oblique asymptote**.

EXAMPLE 9 ■ A Rational Function with a Slant Asymptote

Graph the rational function $r(x) = \frac{x^2 - 4x - 5}{x - 3}$.

SOLUTION

Factor. $y = \frac{(x+1)(x-5)}{x-3}$

x-Intercepts. -1 and 5 , from $x+1=0$ and $x-5=0$

y-Intercept. $\frac{5}{3}$, because $r(0) = \frac{0^2 - 4 \cdot 0 - 5}{0 - 3} = \frac{5}{3}$

Vertical asymptote. $x = 3$, from the zero of the denominator

Behavior near vertical asymptote. $y \rightarrow \infty$ as $x \rightarrow 3^-$ and $y \rightarrow -\infty$ as $x \rightarrow 3^+$

Horizontal asymptote. None, because the degree of the numerator is greater than the degree of the denominator

$$\begin{array}{r}
 x - 1 \\
 x - 3 \overline{) x^2 - 4x - 5} \\
 \underline{x^2 - 3x} \\
 -x - 5 \\
 \underline{-x + 3} \\
 -8
 \end{array}$$

Slant asymptote. Since the degree of the numerator is one more than the degree of the denominator, the function has a slant asymptote. Dividing (see the margin), we obtain

$$r(x) = x - 1 - \frac{8}{x - 3}$$

Thus $y = x - 1$ is the slant asymptote.

Graph. We use the information we have found, together with some additional values, to sketch the graph in Figure 11.

x	y
-2	-1.4
1	4
2	9
4	-5
6	2.33

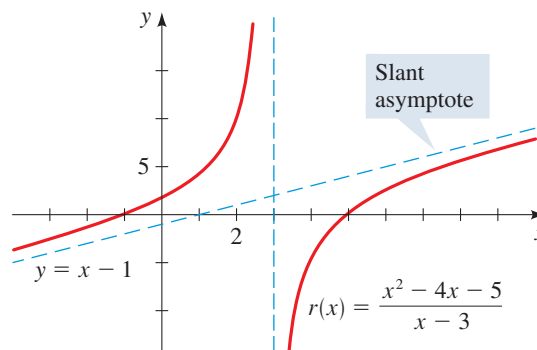


FIGURE 11

Now Try Exercise 69

So far, we have considered only horizontal and slant asymptotes as end behaviors for rational functions. In the next example we graph a function whose end behavior is like that of a parabola.

EXAMPLE 10 ■ End Behavior of a Rational Function

Graph the rational function

$$r(x) = \frac{x^3 - 2x^2 + 3}{x - 2}$$

and describe its end behavior.

SOLUTION

Factor. $y = \frac{(x + 1)(x^2 - 3x + 3)}{x - 2}$

x-Intercept. -1, from $x + 1 = 0$ (The other factor in the numerator has no real zeros.)

y-Intercept. $-\frac{3}{2}$, because $r(0) = \frac{0^3 - 2 \cdot 0^2 + 3}{0 - 2} = -\frac{3}{2}$

Vertical asymptote. $x = 2$, from the zero of the denominator

Behavior near vertical asymptote. $y \rightarrow -\infty$ as $x \rightarrow 2^-$ and $y \rightarrow \infty$ as $x \rightarrow 2^+$

Horizontal asymptote. None, because the degree of the numerator is greater than the degree of the denominator

End behavior. Dividing (see the margin), we get

$$r(x) = x^2 + \frac{3}{x - 2}$$

This shows that the end behavior of r is like that of the parabola $y = x^2$ because $3/(x - 2)$ is small when $|x|$ is large. That is, $3/(x - 2) \rightarrow 0$ as $x \rightarrow \pm\infty$. This means that the graph of r will be close to the graph of $y = x^2$ for large $|x|$.

$$\begin{array}{r}
 x^2 \\
 x - 2 \overline{) x^3 - 2x^2 + 0x + 3} \\
 \underline{x^3 - 2x^2} \\
 3
 \end{array}$$

Graph. In Figure 12(a) we graph r in a small viewing rectangle; we can see the intercepts, the vertical asymptotes, and the local minimum. In Figure 12(b) we graph r in a larger viewing rectangle; here the graph looks almost like the graph of a parabola. In Figure 12(c) we graph both $y = r(x)$ and $y = x^2$; these graphs are very close to each other except near the vertical asymptote.

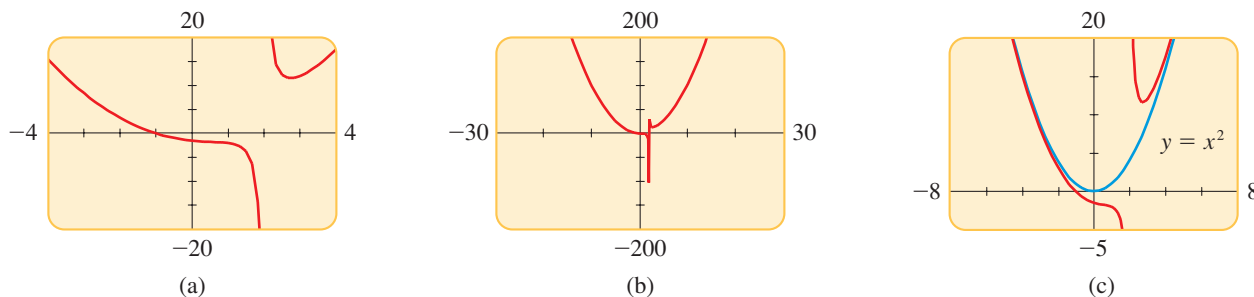


FIGURE 12

$$r(x) = \frac{x^3 - 2x^2 + 3}{x - 2}$$

Now Try Exercise 77

■ Applications

Rational functions occur frequently in scientific applications of algebra. In the next example we analyze the graph of a function from the theory of electricity.

EXAMPLE 11 ■ Electrical Resistance

When two resistors with resistances R_1 and R_2 are connected in parallel, their combined resistance R is given by the formula

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

Suppose that a fixed 8-ohm resistor is connected in parallel with a variable resistor, as shown in Figure 13. If the resistance of the variable resistor is denoted by x , then the combined resistance R is a function of x . Graph R , and give a physical interpretation of the graph.

SOLUTION Substituting $R_1 = 8$ and $R_2 = x$ into the formula gives the function

$$R(x) = \frac{8x}{8 + x}$$

Since resistance cannot be negative, this function has physical meaning only when $x > 0$. The function is graphed in Figure 14(a) using the viewing rectangle $[0, 20]$ by $[0, 10]$. The function has no vertical asymptote when x is restricted to positive values. The combined resistance R increases as the variable resistance x increases. If we widen the viewing rectangle to $[0, 100]$ by $[0, 10]$, we obtain the graph in Figure 14(b). For large x the combined resistance R levels off, getting closer and closer to the horizontal asymptote $R = 8$. No matter how large the variable resistance x , the combined resistance is never greater than 8 ohms.

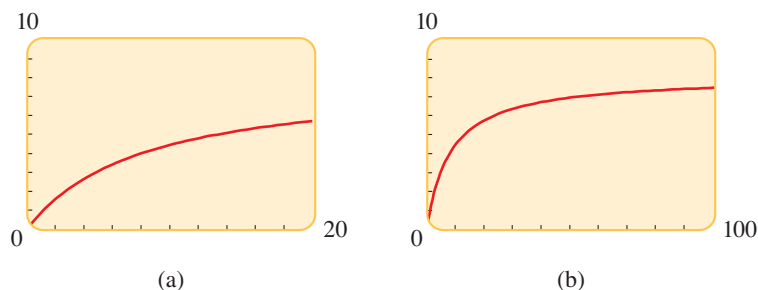


FIGURE 14

$$R(x) = \frac{8x}{8 + x}$$

Now Try Exercise 87

3.6 EXERCISES

CONCEPTS

- If the rational function $y = r(x)$ has the vertical asymptote $x = 2$, then as $x \rightarrow 2^+$, either $y \rightarrow \underline{\hspace{2cm}}$ or $y \rightarrow \underline{\hspace{2cm}}$.
 - If the rational function $y = r(x)$ has the horizontal asymptote $y = 2$, then $y \rightarrow \underline{\hspace{2cm}}$ as $x \rightarrow \pm\infty$.
- 3–6 ■** The following questions are about the rational function
- $$r(x) = \frac{(x+1)(x-2)}{(x+2)(x-3)}$$
- The function r has x -intercepts $\underline{\hspace{2cm}}$ and $\underline{\hspace{2cm}}$.
 - The function r has y -intercept $\underline{\hspace{2cm}}$.
 - The function r has vertical asymptotes $x = \underline{\hspace{2cm}}$ and $x = \underline{\hspace{2cm}}$.
 - The function r has horizontal asymptote $y = \underline{\hspace{2cm}}$.

7–8 ■ True or False?

- Let $r(x) = \frac{x^2 + x}{(x+1)(2x-4)}$. The graph of r has
 - vertical asymptote $x = -1$.
 - vertical asymptote $x = 2$.
 - horizontal asymptote $y = 1$.
 - horizontal asymptote $y = \frac{1}{2}$.
- The graph of a rational function may cross a horizontal asymptote.

SKILLS

- 9–12 ■ Table of Values** A rational function is given. (a) Complete each table for the function. (b) Describe the behavior of the function near its vertical asymptote, based on Tables 1 and 2. (c) Determine the horizontal asymptote, based on Tables 3 and 4.

TABLE 1

x	$r(x)$
1.5	
1.9	
1.99	
1.999	

TABLE 2

x	$r(x)$
2.5	
2.1	
2.01	
2.001	

TABLE 3

x	$r(x)$
10	
50	
100	
1000	

TABLE 4

x	$r(x)$
-10	
-50	
-100	
-1000	

 **9.** $r(x) = \frac{x}{x-2}$

10. $r(x) = \frac{4x+1}{x-2}$

11. $r(x) = \frac{3x-10}{(x-2)^2}$

12. $r(x) = \frac{3x^2+1}{(x-2)^2}$

13–20 ■ Graphing Rational Functions Using Transformations

Use transformations of the graph of $y = 1/x$ to graph the rational function, and state the domain and range, as in Example 2.

13. $r(x) = \frac{1}{x-1}$

14. $r(x) = \frac{1}{x+4}$

 **15.** $s(x) = \frac{3}{x+1}$

16. $s(x) = \frac{-2}{x-2}$

 **17.** $t(x) = \frac{2x-3}{x-2}$

18. $t(x) = \frac{3x-3}{x+2}$

19. $r(x) = \frac{x+2}{x+3}$

20. $r(x) = \frac{2x-9}{x-4}$

21–26 ■ Intercepts of Rational Functions Find the x - and y -intercepts of the rational function.

21. $r(x) = \frac{x-1}{x+4}$

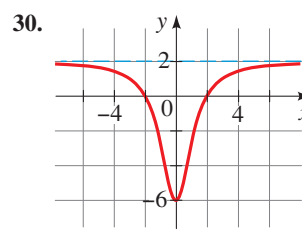
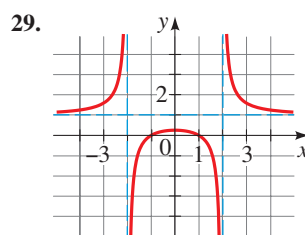
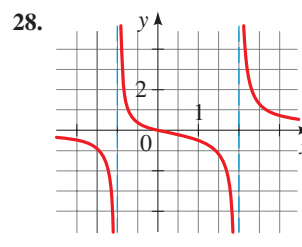
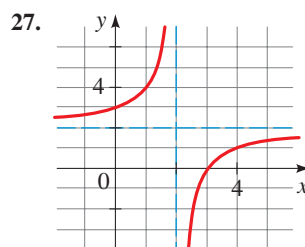
22. $s(x) = \frac{3x}{x-5}$

23. $t(x) = \frac{x^2-x-2}{x-6}$

24. $r(x) = \frac{2}{x^2+3x-4}$

25. $r(x) = \frac{x^2-9}{x^2}$

26. $r(x) = \frac{x^3+8}{x^2+4}$

27–30 ■ Getting Information from a Graph From the graph, determine the x - and y -intercepts and the vertical and horizontal asymptotes.**31–42 ■ Asymptotes** Find all horizontal and vertical asymptotes (if any).

31. $r(x) = \frac{5}{x-2}$

32. $r(x) = \frac{2x-3}{x^2-1}$

$$\text{33. } r(x) = \frac{3x + 1}{4x^2 + 1}$$

$$\text{34. } r(x) = \frac{3x^2 + 5x}{x^4 - 1}$$

$$\text{66. } r(x) = \frac{x^3 - 2x^2 - 3x}{x - 3}$$

$$\text{35. } s(x) = \frac{6x^2 + 1}{2x^2 + x - 1}$$

$$\text{36. } s(x) = \frac{8x^2 + 1}{4x^2 + 2x - 6}$$

$$\text{67. } r(x) = \frac{x^3 - 5x^2 + 3x + 9}{x + 1}$$

[Hint: Check that $x + 1$ is a factor of the numerator.]

$$\text{37. } r(x) = \frac{(x + 1)(2x - 3)}{(x - 2)(4x + 7)}$$

$$\text{38. } r(x) = \frac{(x - 3)(x + 2)}{(5x + 1)(2x - 3)}$$



$$\text{68. } r(x) = \frac{x^2 + 4x - 5}{x^3 + 7x^2 + 10x}$$

$$\text{39. } r(x) = \frac{6x^3 - 2}{2x^3 + 5x^2 + 6x}$$

$$\text{40. } r(x) = \frac{5x^3}{x^3 + 2x^2 + 5x}$$

$$\text{41. } t(x) = \frac{x^2 + 2}{x - 1}$$

$$\text{42. } r(x) = \frac{x^3 + 3x^2}{x^2 - 4}$$

43–62 ■ Graphing Rational Functions Find the intercepts and asymptotes, and then sketch a graph of the rational function and state the domain and range. Use a graphing device to confirm your answer.

$$\text{43. } r(x) = \frac{4x - 4}{x + 2}$$

$$\text{44. } r(x) = \frac{2x + 6}{-6x + 3}$$

$$\text{45. } r(x) = \frac{3x^2 - 12x + 13}{x^2 - 4x + 4}$$

$$\text{46. } r(x) = \frac{-2x^2 - 8x - 9}{x^2 + 4x + 4}$$

$$\text{47. } r(x) = \frac{-x^2 + 8x - 18}{x^2 - 8x + 16}$$

$$\text{48. } r(x) = \frac{x^2 + 2x + 3}{2x^2 + 4x + 2}$$

$$\text{49. } s(x) = \frac{4x - 8}{(x - 4)(x + 1)}$$

$$\text{50. } s(x) = \frac{6}{x^2 - 5x - 6}$$

$$\text{51. } s(x) = \frac{2x - 4}{x^2 + x - 2}$$

$$\text{52. } s(x) = \frac{x + 2}{(x + 3)(x - 1)}$$

$$\text{53. } r(x) = \frac{(x - 1)(x + 2)}{(x + 1)(x - 3)}$$

$$\text{54. } r(x) = \frac{2x^2 + 10x - 12}{x^2 + x - 6}$$

$$\text{55. } r(x) = \frac{2x^2 + 2x - 4}{x^2 + x}$$

$$\text{56. } r(x) = \frac{3x^2 + 6}{x^2 - 2x - 3}$$

$$\text{57. } s(x) = \frac{x^2 - 2x + 1}{x^3 - 3x^2}$$

$$\text{58. } r(x) = \frac{x^2 - x - 6}{x^2 + 3x}$$

$$\text{59. } r(x) = \frac{x^2 - 2x + 1}{x^2 + 2x + 1}$$

$$\text{60. } r(x) = \frac{4x^2}{x^2 - 2x - 3}$$

$$\text{61. } r(x) = \frac{5x^2 + 5}{x^2 + 4x + 4}$$

$$\text{62. } t(x) = \frac{x^3 - x^2}{x^3 - 3x - 2}$$

63–68 ■ Rational Functions with Holes Find the factors that are common in the numerator and the denominator. Then find the intercepts and asymptotes, and sketch a graph of the rational function. State the domain and range of the function.

$$\text{63. } r(x) = \frac{x^2 + 4x - 5}{x^2 + x - 2}$$

$$\text{64. } r(x) = \frac{x^2 + 3x - 10}{(x + 1)(x - 3)(x + 5)}$$

$$\text{65. } r(x) = \frac{x^2 - 2x - 3}{x + 1}$$

69–76 ■ Slant Asymptotes Find the slant asymptote and the vertical asymptotes, and sketch a graph of the function.

$$\text{69. } r(x) = \frac{x^2}{x - 2}$$

$$\text{70. } r(x) = \frac{x^2 + 2x}{x - 1}$$

$$\text{71. } r(x) = \frac{x^2 - 2x - 8}{x}$$

$$\text{72. } r(x) = \frac{3x - x^2}{2x - 2}$$

$$\text{73. } r(x) = \frac{x^2 + 5x + 4}{x - 3}$$

$$\text{74. } r(x) = \frac{x^3 + 4}{2x^2 + x - 1}$$

$$\text{75. } r(x) = \frac{x^3 + x^2}{x^2 - 4}$$

$$\text{76. } r(x) = \frac{2x^3 + 2x}{x^2 - 1}$$

SKILLS Plus



77–80 ■ End Behavior Graph the rational function f , and determine all vertical asymptotes from your graph. Then graph f and g in a sufficiently large viewing rectangle to show that they have the same end behavior.

$$\text{77. } f(x) = \frac{2x^2 + 6x + 6}{x + 3}, \quad g(x) = 2x$$

$$\text{78. } f(x) = \frac{-x^3 + 6x^2 - 5}{x^2 - 2x}, \quad g(x) = -x + 4$$

$$\text{79. } f(x) = \frac{x^3 - 2x^2 + 16}{x - 2}, \quad g(x) = x^2$$

$$\text{80. } f(x) = \frac{-x^4 + 2x^3 - 2x}{(x - 1)^2}, \quad g(x) = 1 - x^2$$



81–86 ■ End Behavior Graph the rational function, and find all vertical asymptotes, x - and y -intercepts, and local extrema, correct to the nearest tenth. Then use long division to find a polynomial that has the same end behavior as the rational function, and graph both functions in a sufficiently large viewing rectangle to verify that the end behaviors of the polynomial and the rational function are the same.

$$\text{81. } y = \frac{2x^2 - 5x}{2x + 3}$$

$$\text{82. } y = \frac{x^4 - 3x^3 + x^2 - 3x + 3}{x^2 - 3x}$$

$$\text{83. } y = \frac{x^5}{x^3 - 1}$$

$$\text{84. } y = \frac{x^4}{x^2 - 2}$$

$$\text{85. } r(x) = \frac{x^4 - 3x^3 + 6}{x - 3}$$

$$\text{86. } r(x) = \frac{4 + x^2 - x^4}{x^2 - 1}$$

APPLICATIONS

-  **87. Population Growth** Suppose that the rabbit population on Mr. Jenkins' farm follows the formula

$$p(t) = \frac{3000t}{t + 1}$$

where $t \geq 0$ is the time (in months) since the beginning of the year.

- (a) Draw a graph of the rabbit population.
(b) What eventually happens to the rabbit population?

-  **88. Drug Concentration** After a certain drug is injected into a patient, the concentration c of the drug in the bloodstream is monitored. At time $t \geq 0$ (in minutes since the injection) the concentration (in mg/L) is given by

$$c(t) = \frac{30t}{t^2 + 2}$$

- (a) Draw a graph of the drug concentration.
(b) What eventually happens to the concentration of drug in the bloodstream?

-  **89. Drug Concentration** A drug is administered to a patient, and the concentration of the drug in the bloodstream is monitored. At time $t \geq 0$ (in hours since giving the drug) the concentration (in mg/L) is given by

$$c(t) = \frac{5t}{t^2 + 1}$$

Graph the function c with a graphing device.

- (a) What is the highest concentration of drug that is reached in the patient's bloodstream?
(b) What happens to the drug concentration after a long period of time?
(c) How long does it take for the concentration to drop below 0.3 mg/L?

-  **90. Flight of a Rocket** Suppose a rocket is fired upward from the surface of the earth with an initial velocity v (measured in meters per second). Then the maximum height h (in meters) reached by the rocket is given by the function

$$h(v) = \frac{Rv^2}{2gR - v^2}$$

where $R = 6.4 \times 10^6$ m is the radius of the earth and $g = 9.8$ m/s² is the acceleration due to gravity. Use a graphing device to draw a graph of the function h . (Note that h and v must both be positive, so the viewing rectangle need not contain negative values.) What does the vertical asymptote represent physically?

-  **91. The Doppler Effect** As a train moves toward an observer (see the figure), the pitch of its whistle sounds higher to the observer than it would if the train were at rest, because the crests of the sound waves are compressed closer together. This phenomenon is called the *Doppler effect*. The observed pitch P is a function of the speed v of the train and is given by

$$P(v) = P_0 \left(\frac{s_0}{s_0 - v} \right)$$

where P_0 is the actual pitch of the whistle at the source and

$s_0 = 332$ m/s is the speed of sound in air. Suppose that a train has a whistle pitched at $P_0 = 440$ Hz. Graph the function $y = P(v)$ using a graphing device. How can the vertical asymptote of this function be interpreted physically?

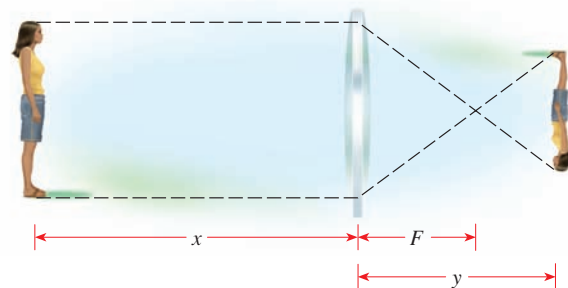


-  **92. Focusing Distance** For a camera with a lens of fixed focal length F to focus on an object located a distance x from the lens, the film must be placed a distance y behind the lens, where F , x , and y are related by

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{F}$$

(See the figure.) Suppose the camera has a 55-mm lens ($F = 55$).

- (a) Express y as a function of x , and graph the function.
(b) What happens to the focusing distance y as the object moves far away from the lens?
(c) What happens to the focusing distance y as the object moves close to the lens?



DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

-  **93. DISCUSS: Constructing a Rational Function from Its Asymptotes** Give an example of a rational function that has vertical asymptote $x = 3$. Now give an example of one that has vertical asymptote $x = 3$ and horizontal asymptote $y = 2$. Now give an example of a rational function with vertical asymptotes $x = 1$ and $x = -1$, horizontal asymptote $y = 0$, and x -intercept 4.

- 94. DISCUSS: A Rational Function with No Asymptote** Explain how you can tell (without graphing it) that the function

$$r(x) = \frac{x^6 + 10}{x^4 + 8x^2 + 15}$$

has no x -intercept and no horizontal, vertical, or slant asymptote. What is its end behavior?"