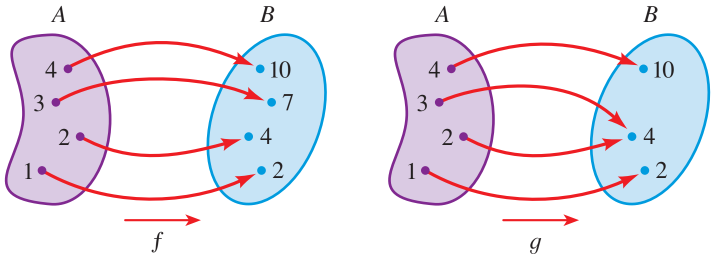
2.8 ONE-TO-ONE FUNCTIONS AND THEIR INVERSES

■ One-to-One Functions ■ The Inverse of a Function ■ Finding the Inverse of a Function ■ Graphing the Inverse of a Function ■ Applications of Inverse Functions

The *inverse* of a function is a rule that acts on the output of the function and produces the corresponding input. So the inverse "undoes" or reverses what the function has done. Not all functions have inverses; those that do are called *one-to-one*.

■ One-to-One Functions

Let's compare the functions f and g whose arrow diagrams are shown in Figure 1. Note that f never takes on the same value twice (any two numbers in A have different images), whereas g does take on the same value twice (both 2 and 3 have the same image, 4). In symbols, $g(2) = g(3)$ but $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. Functions that have this latter property are called *one-to-one*.



f is one-to-one $\quad$ g is not one-to-one

FIGURE 1

DEFINITION OF A ONE-TO-ONE FUNCTION

A function with domain A is called a **one-to-one function** if no two elements of A have the same image, that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2$$

An equivalent way of writing the condition for a one-to-one function is this:

$$\text{If } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

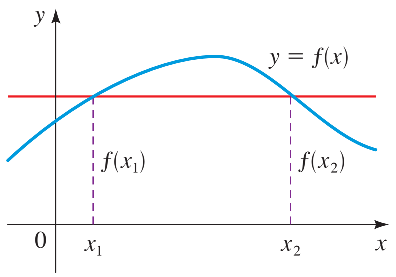


FIGURE 2 This function is not one-to-one because $f(x_1) = f(x_2)$.

If a horizontal line intersects the graph of f at more than one point, then we see from Figure 2 that there are numbers $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. This means that f is not one-to-one. Therefore we have the following geometric method for determining whether a function is one-to-one.

HORIZONTAL LINE TEST

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

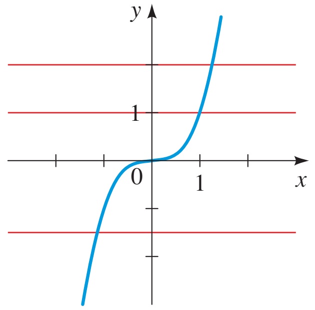


FIGURE 3 $f(x) = x^3$ is one-to-one.

EXAMPLE 1 ■ Deciding Whether a Function Is One-to-One

Is the function $f(x) = x^3$ one-to-one?

SOLUTION 1 If $x_1 \neq x_2$, then $x_1^3 \neq x_2^3$ (two different numbers cannot have the same cube). Therefore $f(x) = x^3$ is one-to-one.

SOLUTION 2 From Figure 3 we see that no horizontal line intersects the graph of $f(x) = x^3$ more than once. Therefore by the Horizontal Line Test, f is one-to-one.

Now Try Exercise 15 ■

Notice that the function f of Example 1 is increasing and is also one-to-one. In fact, it can be proved that *every increasing function and every decreasing function is one-to-one*.

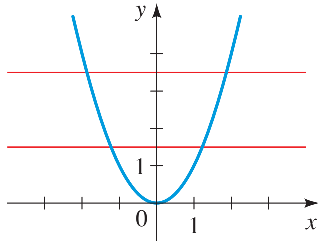


FIGURE 4 $g(x) = x^2$ is not one-to-one.

EXAMPLE 2 ■ Deciding Whether a Function Is One-to-One

Is the function $g(x) = x^2$ one-to-one?

SOLUTION 1 This function is not one-to-one because, for instance,

$$g(1) = 1 \qquad \text{and} \qquad g(-1) = 1$$

so 1 and -1 have the same image.

SOLUTION 2 From Figure 4 we see that there are horizontal lines that intersect the graph of g more than once. Therefore by the Horizontal Line Test, g is not one-to-one.

Now Try Exercise 17 ■

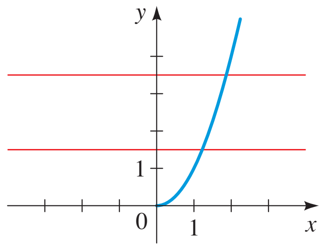


FIGURE 5 $h(x) = x^2$ $(x \geq 0)$ is one-to-one.

Although the function g in Example 2 is not one-to-one, it is possible to restrict its domain so that the resulting function is one-to-one. In fact, if we define

$$h(x) = x^2 \qquad x \geq 0$$

then h is one-to-one, as you can see from Figure 5 and the Horizontal Line Test.

EXAMPLE 3 ■ Showing That a Function Is One-to-One

Show that the function $f(x) = 3x + 4$ is one-to-one.

SOLUTION Suppose there are numbers x_1 and x_2 such that $f(x_1) = f(x_2)$. Then

$$\begin{aligned} 3x_1 + 4 &= 3x_2 + 4 && \text{Suppose } f(x_1) = f(x_2) \\ 3x_1 &= 3x_2 && \text{Subtract 4} \\ x_1 &= x_2 && \text{Divide by 3} \end{aligned}$$

Therefore f is one-to-one.

Now Try Exercise 13 ■

■ The Inverse of a Function

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

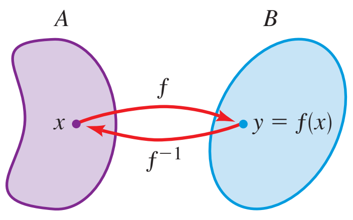


FIGURE 6

DEFINITION OF THE INVERSE OF A FUNCTION

Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

for any y in B.

This definition says that if f takes x to y, then f^{-1} takes y back to x. (If f were not one-to-one, then f^{-1} would not be defined uniquely.) The arrow diagram in Figure 6 indicates that f^{-1} reverses the effect of f. From the definition we have

$$\text{domain of } f^{-1} = \text{range of } f$$
$$\text{range of } f^{-1} = \text{domain of } f$$

EXAMPLE 4 ■ Finding f^{-1} for Specific Values

Don't mistake the -1 in f^{-1} for an exponent.

$$f^{-1}(x) \quad \textit{does not mean} \quad \frac{1}{f(x)}$$

The reciprocal $1/f(x)$ is written as $(f(x))^{-1}$.

If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(5)$, $f^{-1}(7)$, and $f^{-1}(-10)$.

SOLUTION From the definition of f^{-1} we have

$$f^{-1}(5) = 1 \quad \text{because} \quad f(1) = 5$$
$$f^{-1}(7) = 3 \quad \text{because} \quad f(3) = 7$$
$$f^{-1}(-10) = 8 \quad \text{because} \quad f(8) = -10$$

Figure 7 shows how f^{-1} reverses the effect of f in this case.

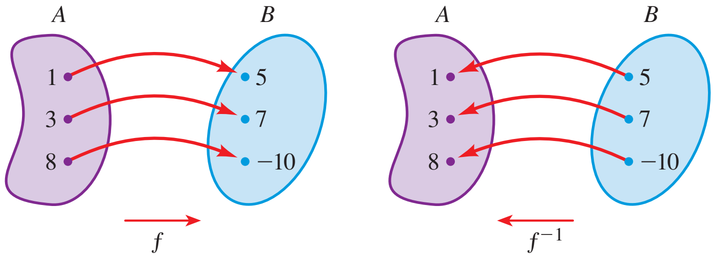


FIGURE 7

Now Try Exercise 25 ■

EXAMPLE 5 ■ Finding Values of an Inverse Function

We can find specific values of an inverse function from a table or graph of the function itself.

(a) The table below gives values of a function h. From the table we see that $h^{-1}(8) = 3$, $h^{-1}(12) = 4$, and $h^{-1}(3) = 6$.

(b) A graph of a function f is shown in Figure 8. From the graph we see that $f^{-1}(5) = 7$ and $f^{-1}(3) = 4$.

x	$h(x)$
2	5
3 ←	8
4 ←	12
5	1
6 ←	3
7	15

Finding values of h^{-1} from a table of h

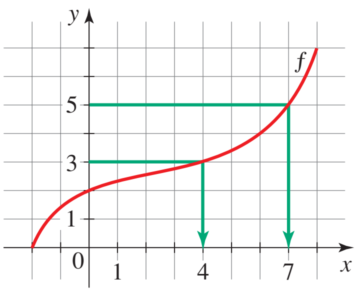


FIGURE 8 Finding values of f^{-1} from a graph of f

Now Try Exercises 29 and 31 ■

By definition the inverse function f^{-1} undoes what f does: If we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started. Similarly, f undoes what f^{-1} does. In general, any function that reverses the effect of f in this way must be the inverse of f. These observations are expressed precisely as follows.

INVERSE FUNCTION PROPERTY

Let f be a one-to-one function with domain A and range B. The inverse function f^{-1} satisfies the following cancellation properties:

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

Conversely, any function f^{-1} satisfying these equations is the inverse of f.

These properties indicate that f is the inverse function of f^{-1}, so we say that f and f^{-1} are *inverses of each other*.

EXAMPLE 6 ■ Verifying That Two Functions Are Inverses

Show that $f(x) = x^3$ and $g(x) = x^{1/3}$ are inverses of each other.

SOLUTION Note that the domain and range of both f and g are $\mathbb{R}$. We have

$$g(f(x)) = g(x^3) = (x^3)^{1/3} = x$$

$$f(g(x)) = f(x^{1/3}) = (x^{1/3})^3 = x$$

So by the Property of Inverse Functions, f and g are inverses of each other. These equations simply say that the cube function and the cube root function, when composed, cancel each other.

Now Try Exercise 39 ■

■ Finding the Inverse of a Function

Now let's examine how we compute inverse functions. We first observe from the definition of f^{-1} that

$$y = f(x) \quad \Leftrightarrow \quad f^{-1}(y) = x$$

So if $y = f(x)$ and if we are able to solve this equation for x in terms of y, then we must have $x = f^{-1}(y)$. If we then interchange x and y, we have $y = f^{-1}(x)$, which is the desired equation.

HOW TO FIND THE INVERSE OF A ONE-TO-ONE FUNCTION

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Note that Steps 2 and 3 can be reversed. In other words, we can interchange x and y first and then solve for y in terms of x.

In Example 7 note how f^{-1} reverses the effect of f. The function f is the rule "Multiply by 3, then subtract 2," whereas f^{-1} is the rule "Add 2, then divide by 3."

EXAMPLE 7 ■ Finding the Inverse of a Function

Find the inverse of the function $f(x) = 3x - 2$.

SOLUTION First we write $y = f(x)$.

$$y = 3x - 2$$

CHECK YOUR ANSWER

We use the Inverse Function Property:

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(3x-2) \\ &= \frac{(3x-2)+2}{3} \\ &= \frac{3x}{3} = x \\ f(f^{-1}(x)) &= f\left(\frac{x+2}{3}\right) \\ &= 3\left(\frac{x+2}{3}\right) - 2 \\ &= x + 2 - 2 = x \quad ✓ \end{aligned}$$

Then we solve this equation for x:

$$3x = y + 2 \qquad \text{Add 2}$$

$$x = \frac{y+2}{3} \qquad \text{Divide by 3}$$

Finally, we interchange x and y:

$$y = \frac{x+2}{3}$$

Therefore, the inverse function is $f^{-1}(x) = \dfrac{x+2}{3}$.

Now Try Exercise 49 ■

EXAMPLE 8 ■ Finding the Inverse of a Function

Find the inverse of the function $f(x) = \dfrac{x^5 - 3}{2}$.

SOLUTION We first write $y = (x^5 - 3)/2$ and solve for x.

$$\begin{aligned} y &= \frac{x^5-3}{2} && \text{Equation defining function} \\ 2y &= x^5 - 3 && \text{Multiply by 2} \\ x^5 &= 2y + 3 && \text{Add 3 (and switch sides)} \\ x &= (2y+3)^{1/5} && \text{Take fifth root of each side} \end{aligned}$$

Then we interchange x and y to get $y = (2x + 3)^{1/5}$. Therefore the inverse function is $f^{-1}(x) = (2x + 3)^{1/5}$.

Now Try Exercise 61 ■

In Example 8 note how f^{-1} reverses the effect of f. The function f is the rule "Take the fifth power, subtract 3, then divide by 2," whereas f^{-1} is the rule "Multiply by 2, add 3, then take the fifth root."

CHECK YOUR ANSWER

We use the Inverse Function Property:

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}\left(\frac{x^5-3}{2}\right) \\ &= \left[2\left(\frac{x^5-3}{2}\right) + 3\right]^{1/5} \\ &= (x^5 - 3 + 3)^{1/5} \\ &= (x^5)^{1/5} = x \\ f(f^{-1}(x)) &= f((2x+3)^{1/5}) \\ &= \frac{[(2x+3)^{1/5}]^5 - 3}{2} \\ &= \frac{2x+3-3}{2} \\ &= \frac{2x}{2} = x \quad ✓ \end{aligned}$$

A **rational function** is a function defined by a rational expression. In the next example we find the inverse of a rational function.

EXAMPLE 9 ■ Finding the Inverse of a Rational Function

Find the inverse of the function $f(x) = \dfrac{2x+3}{x-1}$.

SOLUTION We first write $y = (2x + 3)/(x - 1)$ and solve for x.

$$\begin{aligned} y &= \frac{2x+3}{x-1} && \text{Equation defining function} \\ y(x-1) &= 2x + 3 && \text{Multiply by } x - 1 \\ yx - y &= 2x + 3 && \text{Expand} \\ yx - 2x &= y + 3 && \text{Bring } x\text{-terms to LHS} \\ x(y-2) &= y + 3 && \text{Factor } x \\ x &= \frac{y+3}{y-2} && \text{Divide by } y - 2 \end{aligned}$$

Therefore the inverse function is $f^{-1}(x) = \dfrac{x+3}{x-2}$.

Now Try Exercise 55 ■

Rational functions are studied in Section 3.6.

Graphing the Inverse of a Function

The principle of interchanging x and y to find the inverse function also gives us a method for obtaining the graph of f^{-1} from the graph of f. If $f(a) = b$, then $f^{-1}(b) = a$. Thus the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from the point (a, b) by reflecting in the line $y = x$ (see Figure 9). Therefore, as Figure 10 illustrates, the following is true.

> The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.

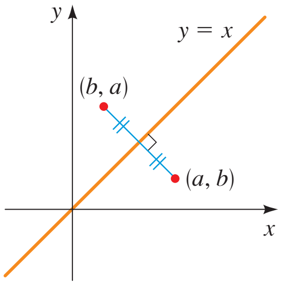


FIGURE 9

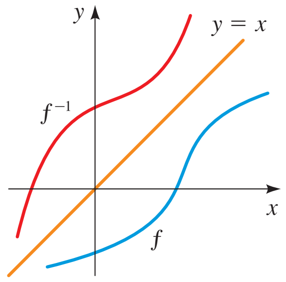


FIGURE 10

EXAMPLE 10 ■ Graphing the Inverse of a Function

(a) Sketch the graph of $f(x) = \sqrt{x - 2}$.

(b) Use the graph of f to sketch the graph of f^{-1}.

(c) Find an equation for f^{-1}.

SOLUTION

(a) Using the transformations from Section 2.6, we sketch the graph of $y = \sqrt{x - 2}$ by plotting the graph of the function $y = \sqrt{x}$ (Example 1(c) in Section 2.2) and shifting it to the right 2 units.

(b) The graph of f^{-1} is obtained from the graph of f in part (a) by reflecting it in the line $y = x$, as shown in Figure 11.

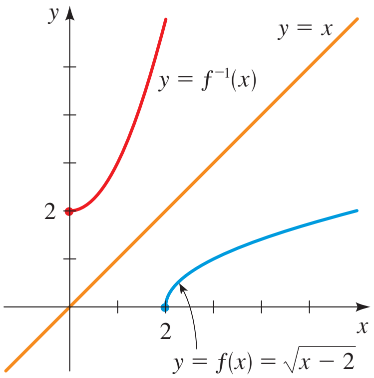


FIGURE 11

(c) Solve $y = \sqrt{x - 2}$ for x, noting that $y \geq 0$.

$$\sqrt{x - 2} = y$$

$$x - 2 = y^2 \qquad \text{Square each side}$$

$$x = y^2 + 2 \qquad y \geq 0 \qquad \text{Add 2}$$

Interchange x and y, as follows:

$$y = x^2 + 2 \qquad x \geq 0$$

Thus $$f^{-1}(x) = x^2 + 2 \qquad x \geq 0$$

This expression shows that the graph of f^{-1} is the right half of the parabola $y = x^2 + 2$, and from the graph shown in Figure 11 this seems reasonable.

Now Try Exercise 73 ■

In Example 10 note how f^{-1} reverses the effect of f. The function f is the rule "Subtract 2, then take the square root," whereas f^{-1} is the rule "Square, then add 2."

Applications of Inverse Functions

When working with functions that model real-world situations, we name the variables using letters that suggest the quantity being modeled. For instance we may use t for time, d for distance, V for volume, and so on. When using inverse functions, we follow this convention. For example, suppose that the variable R is a function of the variable N, say, $R = f(N)$. Then $f^{-1}(R) = N$. So the function f^{-1} defines N as a function of R.

EXAMPLE 11 ■ An Inverse Function

At a local pizza parlor the daily special is \$12 for a plain cheese pizza plus \$2 for each additional topping.

(a) Find a function f that models the price of a pizza with n toppings.

(b) Find the inverse of the function f. What does f^{-1} represent?

(c) If a pizza costs \$22, how many toppings does it have?

SOLUTION Note that the price p of a pizza is a function of the number n of toppings.

(a) The price of a pizza with n toppings is given by the function

$$f(n) = 12 + 2n$$

(b) To find the inverse function, we first write $p = f(n)$, where we use the letter p instead of our usual y because $f(n)$ is the price of the pizza. We have

$$p = 12 + 2n$$

Next we solve for n:

$$\begin{aligned} p &= 12 + 2n \\ p - 12 &= 2n \\ n &= \frac{p - 12}{2} \end{aligned}$$

So $n = f^{-1}(p) = \dfrac{p-12}{2}$. The function f^{-1} gives the number n of toppings for a pizza with price p.

(c) We have $n = f^{-1}(22) = (22 - 12)/2 = 5$. So the pizza has five toppings.

Now Try Exercise 93

2.8 EXERCISES

CONCEPTS

1. A function f is one-to-one if different inputs produce ________ outputs. You can tell from the graph that a function is one-to-one by using the ______________ Test.

2. (a) For a function to have an inverse, it must be ________. So which one of the following functions has an inverse?

$$f(x) = x^2 \qquad g(x) = x^3$$

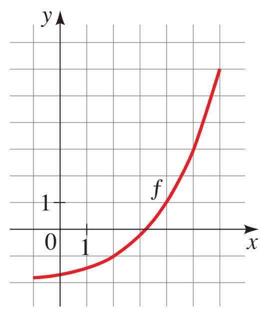
(b) What is the inverse of the function that you chose in part (a)?

3. A function f has the following verbal description: "Multiply by 3, add 5, and then take the third power of the result."

(a) Write a verbal description for f^{-1}.

(b) Find algebraic formulas that express f and f^{-1} in terms of the input x.

4. A graph of a function f is given. Does f have an inverse? If so, find $f^{-1}(1) =$ ________ and $f^{-1}(3) =$ ________.



5. If the point $(3, 4)$ is on the graph of the function f, then the point (____, ____) is on the graph of f^{-1}.

6. *True or false?*

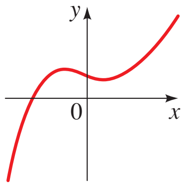
(a) If f has an inverse, then $f^{-1}(x)$ is always the same as $\dfrac{1}{f(x)}$.

(b) If f has an inverse, then $f^{-1}(f(x)) = x$.

SKILLS

7–12 ■ One-to-One Function? A graph of a function f is given. Determine whether f is one-to-one.

7.



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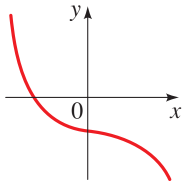

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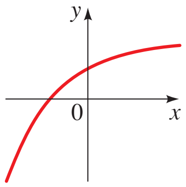

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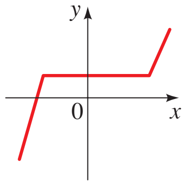

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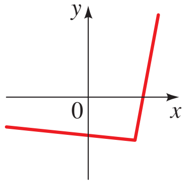

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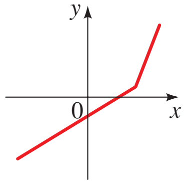


13–24 ■ One-to-One Function? Determine whether the function is one-to-one.

13. $f(x) = -2x + 4$

14. $f(x) = 3x - 2$

15. $g(x) = \sqrt{x}$

16. $g(x) = |x|$

17. $h(x) = x^2 - 2x$

18. $h(x) = x^3 + 8$

19. $f(x) = x^4 + 5$

20. $f(x) = x^4 + 5, \quad 0 \le x \le 2$

21. $r(t) = t^6 - 3, \quad 0 \le t \le 5$

22. $r(t) = t^4 - 1$

23. $f(x) = \dfrac{1}{x^2}$

24. $f(x) = \dfrac{1}{x}$

25–28 ■ Finding Values of an Inverse Function Assume that f is a one-to-one function.

25. (a) If $f(2) = 7$, find $f^{-1}(7)$.

(b) If $f^{-1}(3) = -1$, find $f(-1)$.

26. (a) If $f(5) = 18$, find $f^{-1}(18)$.

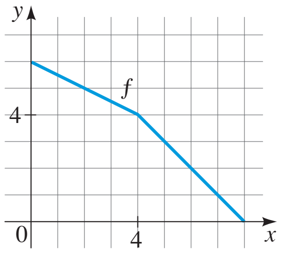
(b) If $f^{-1}(4) = 2$, find $f(2)$.

27. If $f(x) = 5 - 2x$, find $f^{-1}(3)$.

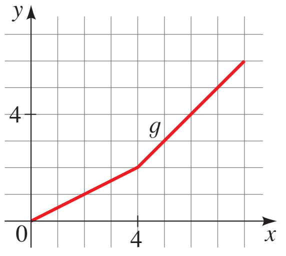
28. If $g(x) = x^2 + 4x$ with $x \ge -2$, find $g^{-1}(5)$.

29–30 ■ Finding Values of an Inverse from a Graph A graph of a function is given. Use the graph to find the indicated values.

29. (a) $f^{-1}(2)$ **(b)** $f^{-1}(5)$ **(c)** $f^{-1}(6)$



30. (a) $g^{-1}(2)$ **(b)** $g^{-1}(5)$ **(c)** $g^{-1}(6)$



31–36 ■ Finding Values of an Inverse Using a Table A table of values for a one-to-one function is given. Find the indicated values.

31. $f^{-1}(5)$

32. $f^{-1}(0)$

33. $f^{-1}(f(1))$

34. $f(f^{-1}(6))$

35. $f^{-1}(f^{-1}(1))$

36. $f^{-1}(f^{-1}(0))$

x	1	2	3	4	5	6
$f(x)$	4	6	2	5	0	1

37–48 ■ Inverse Function Property Use the Inverse Function Property to show that f and g are inverses of each other.

37. $f(x) = x - 6; \quad g(x) = x + 6$

38. $f(x) = 3x; \quad g(x) = \dfrac{x}{3}$

39. $f(x) = 3x + 4; \quad g(x) = \dfrac{x - 4}{3}$

40. $f(x) = 2 - 5x; \quad g(x) = \dfrac{2 - x}{5}$

41. $f(x) = \dfrac{1}{x}; \quad g(x) = \dfrac{1}{x}$

42. $f(x) = x^5; \quad g(x) = \sqrt[5]{x}$

43. $f(x) = x^2 - 9, \quad x \ge 0; \quad g(x) = \sqrt{x + 9}, \quad x \ge -9$

44. $f(x) = x^3 + 1; \quad g(x) = (x - 1)^{1/3}$

45. $f(x) = \dfrac{1}{x - 1}; \quad g(x) = \dfrac{1}{x} + 1$

46. $f(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2;$
$g(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2$

47. $f(x) = \dfrac{x + 2}{x - 2}; \quad g(x) = \dfrac{2x + 2}{x - 1}$

48. $f(x) = \dfrac{x - 5}{3x + 4}; \quad g(x) = \dfrac{5 + 4x}{1 - 3x}$

49–70 ■ Finding Inverse Functions Find the inverse function of f.

49. $f(x) = 3x + 5$

50. $f(x) = 7 - 5x$

51. $f(x) = 5 - 4x^3$

52. $f(x) = 3x^3 + 8$

53. $f(x) = \dfrac{1}{x + 2}$

54. $f(x) = \dfrac{x - 2}{x + 2}$

55. $f(x) = \dfrac{x}{x + 4}$

56. $f(x) = \dfrac{3x}{x - 2}$

57. $f(x) = \dfrac{2x + 5}{x - 7}$

58. $f(x) = \dfrac{4x - 2}{3x + 1}$

59. $f(x) = \dfrac{2x + 3}{1 - 5x}$

60. $f(x) = \dfrac{3 - 4x}{8x - 1}$

61. $f(x) = 4 - x^2, \quad x \ge 0$

62. $f(x) = x^2 + x, \quad x \ge -\frac{1}{2}$

63. $f(x) = x^6, \quad x \ge 0$

64. $f(x) = \dfrac{1}{x^2}, \quad x > 0$

65. $f(x) = \dfrac{2 - x^3}{5}$

66. $f(x) = (x^5 - 6)^7$

67. $f(x) = \sqrt{5 + 8x}$

68. $f(x) = 2 + \sqrt{3 + x}$

69. $f(x) = 2 + \sqrt[3]{x}$

70. $f(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2$

71–74 ■ Graph of an Inverse Function A function f is given. **(a)** Sketch the graph of f. **(b)** Use the graph of f to sketch the graph of f^{-1}. **(c)** Find f^{-1}.

71. $f(x) = 3x - 6$

72. $f(x) = 16 - x^2, \quad x \ge 0$

73. $f(x) = \sqrt{x + 1}$

74. $f(x) = x^3 - 1$

75–80 ■ One-to-One Functions from a Graph Draw the graph of f, and use it to determine whether the function is one-to-one.

75. $f(x) = x^3 - x$

76. $f(x) = x^3 + x$

77. $f(x) = \dfrac{x + 12}{x - 6}$

78. $f(x) = \sqrt{x^3 - 4x + 1}$

79. $f(x) = |x| - |x - 6|$

80. $f(x) = x \cdot |x|$

81–84 ■ Finding Inverse Functions A one-to-one function is given. **(a)** Find the inverse of the function. **(b)** Graph both the function and its inverse on the same screen to verify that the graphs are reflections of each other in the line $y = x$.

81. $f(x) = 2 + x$

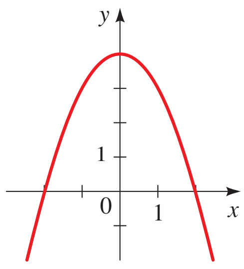
82. $f(x) = 2 - \frac{1}{2}x$

83. $g(x) = \sqrt{x + 3}$

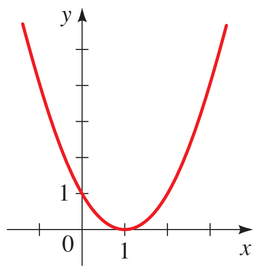
84. $g(x) = x^2 + 1, \quad x \ge 0$

85–88 ■ Restricting the Domain The given function is not one-to-one. Restrict its domain so that the resulting function *is* one-to-one. Find the inverse of the function with the restricted domain. (There is more than one correct answer.)

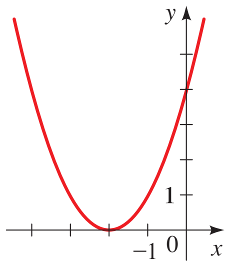
85. $f(x) = 4 - x^2$



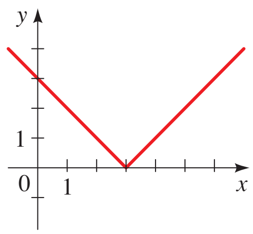
86. $g(x) = (x - 1)^2$



87. $h(x) = (x + 2)^2$

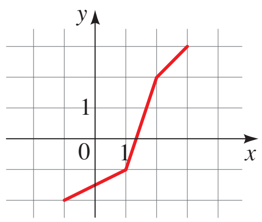


88. $k(x) = |x - 3|$

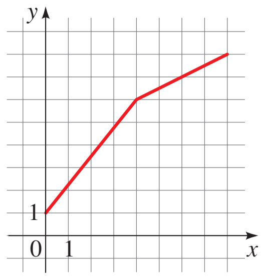


89–90 ■ Graph of an Inverse Function Use the graph of f to sketch the graph of f^{-1}.

89.



90.



SKILLS Plus

91–92 ■ Functions That Are Their Own Inverse If a function f is its own inverse, then the graph of f is symmetric about the line $y = x$. **(a)** Graph the given function. **(b)** Does the graph indicate that f and f^{-1} are the same function? **(c)** Find the function f^{-1}. Use your result to verify your answer to part (b).

91. $f(x) = \dfrac{1}{x}$

92. $f(x) = \dfrac{x + 3}{x - 1}$

APPLICATIONS

93. Pizza Cost Marcello's Pizza charges a base price of \$16 for a large pizza plus \$1.50 for each additional topping.

(a) Find a function f that models the price of a pizza with n toppings.

(b) Find the inverse of the function f. What does f^{-1} represent?

(c) If a pizza costs \$25, how many toppings does it have?

94. Fee for Service For his services, a private investigator requires a \$500 retainer fee plus \$80 per hour. Let x represent the number of hours the investigator spends working on a case.

(a) Find a function f that models the investigator's fee as a function of x.

(b) Find f^{-1}. What does f^{-1} represent?

(c) Find $f^{-1}(1220)$. What does your answer represent?

95. Torricelli's Law A tank holds 100 gallons of water, which drains from a leak at the bottom, causing the tank to empty in 40 minutes. According to Torricelli's Law, the volume V of water remaining in the tank after t min is given by the function

$$V = f(t) = 100\left(1 - \frac{t}{40}\right)^2$$

(a) Find f^{-1}. What does f^{-1} represent?

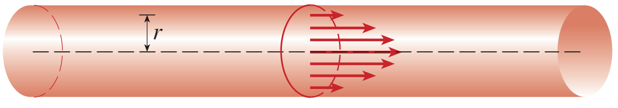
(b) Find $f^{-1}(15)$. What does your answer represent?

96. Blood Flow As blood moves through a vein or artery, its velocity v is greatest along the central axis and decreases as the distance r from the central axis increases (see the figure below). For an artery with radius 0.5 cm, v (in cm/s) is given as a function of r (in cm) by

$$v = g(r) = 18{,}500(0.25 - r^2)$$

(a) Find g^{-1}. What does g^{-1} represent?

(b) Find $g^{-1}(30)$. What does your answer represent?



97. Demand Function The amount of a commodity that is sold is called the *demand* for the commodity. The demand D for a certain commodity is a function of the price given by

$$D = f(p) = -3p + 150$$

(a) Find f^{-1}. What does f^{-1} represent?

(b) Find $f^{-1}(30)$. What does your answer represent?

98. Temperature Scales The relationship between the Fahrenheit (F) and Celsius (C) scales is given by

$$F = g(C) = \tfrac{9}{5}C + 32$$

(a) Find g^{-1}. What does g^{-1} represent?

(b) Find $g^{-1}(86)$. What does your answer represent?

99. Exchange Rates The relative value of currencies fluctuates every day. When this problem was written, one Canadian dollar was worth 0.9766 U.S. dollars.

(a) Find a function f that gives the U.S. dollar value $f(x)$ of x Canadian dollars.

(b) Find f^{-1}. What does f^{-1} represent?

(c) How much Canadian money would \$12,250 in U.S. currency be worth?

100. Income Tax In a certain country the tax on incomes less than or equal to €20,000 is 10%. For incomes that are more than €20,000 the tax is €2000 plus 20% of the amount over €20,000.

(a) Find a function f that gives the income tax on an income x. Express f as a piecewise defined function.

(b) Find f^{-1}. What does f^{-1} represent?

(c) How much income would require paying a tax of €10,000?

101. Multiple Discounts A car dealership advertises a 15% discount on all its new cars. In addition, the manufacturer offers a \$1000 rebate on the purchase of a new car. Let x represent the sticker price of the car.

(a) Suppose that only the 15% discount applies. Find a function f that models the purchase price of the car as a function of the sticker price x.

(b) Suppose that only the \$1000 rebate applies. Find a function g that models the purchase price of the car as a function of the sticker price x.

(c) Find a formula for $H = f \circ g$.

(d) Find H^{-1}. What does H^{-1} represent?

(e) Find $H^{-1}(13{,}000)$. What does your answer represent?

DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

102. DISCUSS: Determining When a Linear Function Has an Inverse For the linear function $f(x) = mx + b$ to be one-to-one, what must be true about its slope? If it is one-to-one, find its inverse. Is the inverse linear? If so, what is its slope?

103. DISCUSS: Finding an Inverse "in Your Head" In the margin notes in this section we pointed out that the inverse of a function can be found by simply reversing the operations that make up the function. For instance, in Example 7 we saw that the inverse of

$$f(x) = 3x - 2 \qquad \text{is} \qquad f^{-1}(x) = \frac{x+2}{3}$$

because the "reverse" of "Multiply by 3 and subtract 2" is "Add 2 and divide by 3." Use the same procedure to find the inverse of the following functions.

(a) $f(x) = \dfrac{2x+1}{5}$ **(b)** $f(x) = 3 - \dfrac{1}{x}$

(c) $f(x) = \sqrt{x^3 + 2}$ **(d)** $f(x) = (2x - 5)^3$

Now consider another function:

$$f(x) = x^3 + 2x + 6$$

Is it possible to use the same sort of simple reversal of operations to find the inverse of this function? If so, do it. If not, explain what is different about this function that makes this task difficult.

104. PROVE: The Identity Function The function $I(x) = x$ is called the **identity function**. Show that for any function f we have $f \circ I = f$, $I \circ f = f$, and $f \circ f^{-1} = f^{-1} \circ f = I$. (This means that the identity function I behaves for functions and composition just the way the number 1 behaves for real numbers and multiplication.)