2.4 AVERAGE RATE OF CHANGE OF A FUNCTION

■ Average Rate of Change ■ Linear Functions Have Constant Rate of Change

Functions are often used to model changing quantities. In this section we learn how to find the rate at which the values of a function change as the input variable changes.

■ Average Rate of Change

We are all familiar with the concept of speed: If you drive a distance of 120 miles in 2 hours, then your average speed, or rate of travel, is $\frac{120\text{ mi}}{2\text{ h}} = 60$ mi/h. Now suppose you take a car trip and record the distance that you travel every few minutes. The distance s you have traveled is a function of the time t:

$$s(t) = \text{total distance traveled at time } t$$

We graph the function s as shown in Figure 1. The graph shows that you have traveled a total of 50 miles after 1 hour, 75 miles after 2 hours, 140 miles after 3 hours, and so on. To find your *average* speed between any two points on the trip, we divide the distance traveled by the time elapsed.

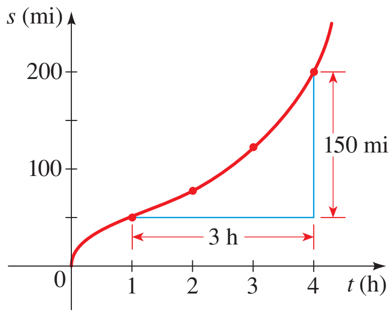


FIGURE 1 Average speed

Let's calculate your average speed between 1:00 P.M. and 4:00 P.M. The time elapsed is $4 - 1 = 3$ hours. To find the distance you traveled, we subtract the distance at 1:00 P.M. from the distance at 4:00 P.M., that is, $200 - 50 = 150$ mi. Thus your average speed is

$$\text{average speed} = \frac{\text{distance traveled}}{\text{time elapsed}} = \frac{150\text{ mi}}{3\text{ h}} = 50\text{ mi/h}$$

The average speed that we have just calculated can be expressed by using function notation:

$$\text{average speed} = \frac{s(4) - s(1)}{4 - 1} = \frac{200 - 50}{3} = 50\text{ mi/h}$$

Note that the average speed is different over different time intervals. For example, between 2:00 P.M. and 3:00 P.M. we find that

$$\text{average speed} = \frac{s(3) - s(2)}{3 - 2} = \frac{140 - 75}{1} = 65\text{ mi/h}$$

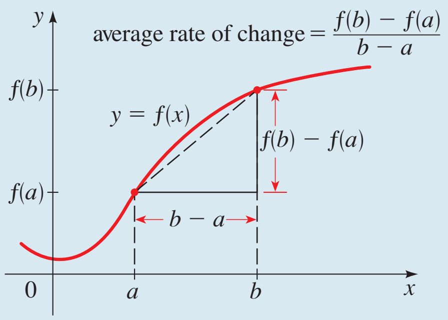
Finding average rates of change is important in many contexts. For instance, we might be interested in knowing how quickly the air temperature is dropping as a storm approaches or how fast revenues are increasing from the sale of a new product. So we need to know how to determine the average rate of change of the functions that model these quantities. In fact, the concept of average rate of change can be defined for any function.

AVERAGE RATE OF CHANGE

The **average rate of change** of the function $y = f(x)$ between $x = a$ and $x = b$ is

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

The average rate of change is the slope of the **secant line** between $x = a$ and $x = b$ on the graph of f, that is, the line that passes through $(a, f(a))$ and $(b, f(b))$.



In the expression for average rate of change, the numerator $f(b) - f(a)$ is the net change in the value of f between $x = a$ and $x = b$ (see page 151).

EXAMPLE 1 ■ Calculating the Average Rate of Change

For the function $f(x) = (x - 3)^2$, whose graph is shown in Figure 2, find the net change and the average rate of change between the following points:

(a) $x = 1$ and $x = 3$ **(b)** $x = 4$ and $x = 7$

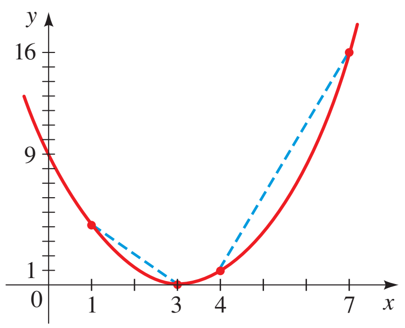


FIGURE 2 $f(x) = (x - 3)^2$

SOLUTION

(a) Net change $= f(3) - f(1)$ — Definition

$= (3 - 3)^2 - (1 - 3)^2$ — Use $f(x) = (x - 3)^2$

$= -4$ — Calculate

$$\text{Average rate of change} = \frac{f(3) - f(1)}{3 - 1} \quad \text{Definition}$$

$$= \frac{-4}{2} = -2 \quad \text{Calculate}$$

(b) Net change $= f(7) - f(4)$ — Definition

$= (7 - 3)^2 - (4 - 3)^2$ — Use $f(x) = (x - 3)^2$

$= 15$ — Calculate

$$\text{Average rate of change} = \frac{f(7) - f(4)}{7 - 4} \quad \text{Definition}$$

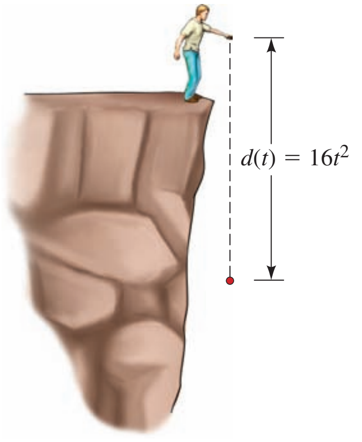
$$= \frac{15}{3} = 5 \quad \text{Calculate}$$

Now Try Exercise 15 ■

EXAMPLE 2 ■ Average Speed of a Falling Object

If an object is dropped from a high cliff or a tall building, then the distance it has fallen after t seconds is given by the function $d(t) = 16t^2$. Find its average speed (average rate of change) over the following intervals:

(a) Between 1 s and 5 s **(b)** Between $t = a$ and $t = a + h$



Function: In t seconds the stone falls $16t^2$ ft.

SOLUTION

(a) Average rate of change $= \dfrac{d(5) - d(1)}{5 - 1}$ Definition

$= \dfrac{16(5)^2 - 16(1)^2}{5 - 1}$ Use $d(t) = 16t^2$

$= \dfrac{400 - 16}{4}$ Calculate

$= 96$ ft/s Calculate

(b) Average rate of change $= \dfrac{d(a + h) - d(a)}{(a + h) - a}$ Definition

$= \dfrac{16(a + h)^2 - 16(a)^2}{(a + h) - a}$ Use $d(t) = 16t^2$

$= \dfrac{16(a^2 + 2ah + h^2 - a^2)}{h}$ Expand and factor 16

$= \dfrac{16(2ah + h^2)}{h}$ Simplify numerator

$= \dfrac{16h(2a + h)}{h}$ Factor h

$= 16(2a + h)$ Simplify

Now Try Exercise 19 ■

The average rate of change calculated in Example 2(b) is known as a *difference quotient*. In calculus we use difference quotients to calculate *instantaneous* rates of change. An example of an instantaneous rate of change is the speed shown on the speedometer of your car. This changes from one instant to the next as your car's speed changes.

The graphs in Figure 3 show that if a function is increasing on an interval, then the average rate of change between any two points is positive, whereas if a function is decreasing on an interval, then the average rate of change between any two points is negative.

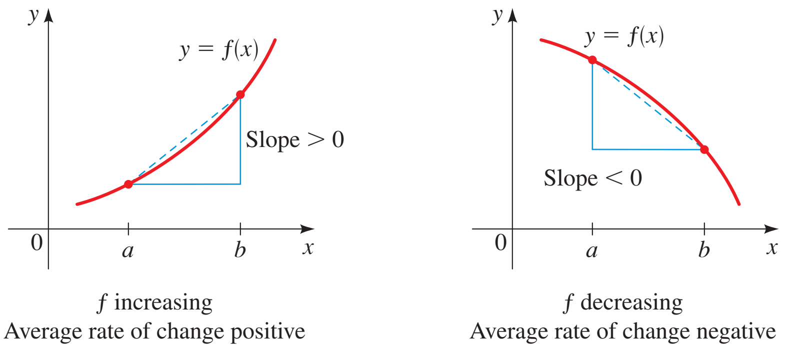


FIGURE 3

EXAMPLE 3 ■ Average Rate of Temperature Change

Time	Temperature (°F)
8:00 A.M.	38
9:00 A.M.	40
10:00 A.M.	44
11:00 A.M.	50
12:00 NOON	56
1:00 P.M.	62
2:00 P.M.	66
3:00 P.M.	67
4:00 P.M.	64
5:00 P.M.	58
6:00 P.M.	55
7:00 P.M.	51

The table in the margin gives the outdoor temperatures observed by a science student on a spring day. Draw a graph of the data, and find the average rate of change of temperature between the following times:

(a) 8:00 A.M. and 9:00 A.M.

(b) 1:00 P.M. and 3:00 P.M.

(c) 4:00 P.M. and 7:00 P.M.

SOLUTION A graph of the temperature data is shown in Figure 4. Let t represent time, measured in hours since midnight (so, for example, 2:00 P.M. corresponds to $t = 14$). Define the function F by

$$F(t) = \text{temperature at time } t$$

Temperature at 9:00 A.M. Temperature at 8:00 A.M.

(a) $$\text{Average rate of change} = \frac{F(9) - F(8)}{9 - 8} = \frac{40 - 38}{9 - 8} = 2$$

The average rate of change was 2°F per hour.

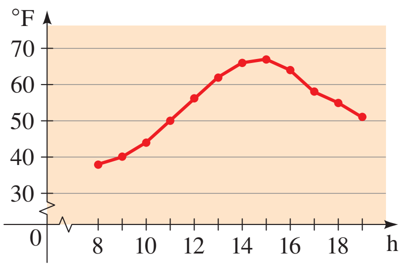


FIGURE 4

(b) $$\text{Average rate of change} = \frac{F(15) - F(13)}{15 - 13} = \frac{67 - 62}{2} = 2.5$$

The average rate of change was 2.5°F per hour.

(c) $$\text{Average rate of change} = \frac{F(19) - F(16)}{19 - 16} = \frac{51 - 64}{3} \approx -4.3$$

The average rate of change was about −4.3°F per hour during this time interval. The negative sign indicates that the temperature was dropping.

Now Try Exercise 31 ■

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DISCOVERY PROJECT

When Rates of Change Change

In the real world, rates of change often themselves change. A statement like "inflation is rising, but at a slower rate" involves a change of a rate of change. When you drive your car, your speed (rate of change of distance) increases when you accelerate and decreases when you decelerate. From Example 4 we see that functions whose graph is a line (linear functions) have constant rate of change. In this project we explore how the shape of a graph corresponds to a changing rate of change. You can find the project at **www.stewartmath.com**.

Linear Functions Have Constant Rate of Change

Recall that a function of the form $f(x) = mx + b$ is a linear function (see page 160). Its graph is a line with slope m. On the other hand, if a function f has constant rate of change, then it must be a linear function. (You are asked to prove these facts in Exercises 51 and 52 in Section 2.5.) In general, the average rate of change of a linear function between any two points is the constant m. In the next example we find the average rate of change for a particular linear function.

EXAMPLE 4 ■ Linear Functions Have Constant Rate of Change

Let $f(x) = 3x - 5$. Find the average rate of change of f between the following points.

(a) $x = 0$ and $x = 1$

(b) $x = 3$ and $x = 7$

(c) $x = a$ and $x = a + h$

What conclusion can you draw from your answers?

SOLUTION

(a) $$\text{Average rate of change} = \frac{f(1) - f(0)}{1 - 0} = \frac{(3\cdot 1 - 5) - (3\cdot 0 - 5)}{1} = \frac{(-2) - (-5)}{1} = 3$$

(b) $$\text{Average rate of change} = \frac{f(7) - f(3)}{7 - 3} = \frac{(3\cdot 7 - 5) - (3\cdot 3 - 5)}{4} = \frac{16 - 4}{4} = 3$$

(c) $$\text{Average rate of change} = \frac{f(a + h) - f(a)}{(a + h) - a} = \frac{[3(a + h) - 5] - [3a - 5]}{h} = \frac{3a + 3h - 5 - 3a + 5}{h} = \frac{3h}{h} = 3$$

It appears that the average rate of change is always 3 for this function. In fact, part (c) proves that the rate of change between any two arbitrary points $x = a$ and $x = a + h$ is 3.

Now Try Exercise 25

2.4 EXERCISES

CONCEPTS

1. If you travel 100 miles in two hours, then your average speed for the trip is

$$\text{average speed} = \frac{\square}{\square} = \underline{\qquad}$$

2. The average rate of change of a function f between $x = a$ and $x = b$ is

$$\text{average rate of change} = \frac{\square}{\square}$$

3. The average rate of change of the function $f(x) = x^2$ between $x = 1$ and $x = 5$ is

$$\text{average rate of change} = \frac{\square}{\square} = \underline{\qquad}$$

4. (a) The average rate of change of a function f between $x = a$ and $x = b$ is the slope of the ________ line between $(a, f(a))$ and $(b, f(b))$.

(b) The average rate of change of the linear function $f(x) = 3x + 5$ between any two points is ________.

5–6 ■ *Yes or No*? If *No*, give a reason.

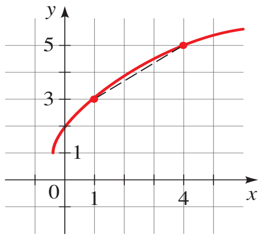
5. **(a)** Is the average rate of change of a function between $x = a$ and $x = b$ the slope of the secant line through $(a, f(a))$ and $(b, f(b))$?
(b) Is the average rate of change of a linear function the same for all intervals?

6. **(a)** Can the average rate of change of an increasing function ever be negative?
(b) If the average rate of change of a function between $x = a$ and $x = b$ is negative, then is the function necessarily decreasing on the interval (a, b)?

SKILLS

7–10 ■ **Net Change and Average Rate of Change** The graph of a function is given. Determine **(a)** the net change and **(b)** the average rate of change between the indicated points on the graph.

7.



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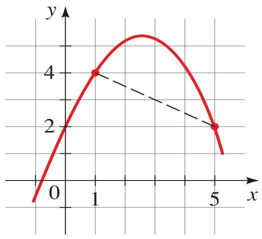

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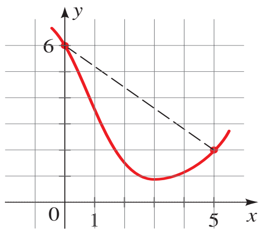

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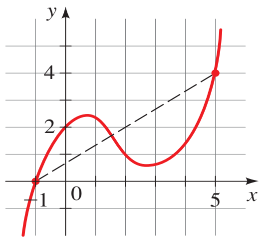


11–24 ■ **Net Change and Average Rate of Change** A function is given. Determine **(a)** the net change and **(b)** the average rate of change between the given values of the variable.

11. $f(x) = 3x - 2;\quad x = 2, x = 3$

12. $r(t) = 3 - \frac{1}{3}t;\quad t = 3, t = 6$

13. $h(t) = -t + \frac{3}{2};\quad t = -4, t = 1$

14. $g(x) = 2 - \frac{2}{3}x;\quad x = -3, x = 2$

15. $h(t) = 2t^2 - t;\quad t = 3, t = 6$

16. $f(z) = 1 - 3z^2;\quad z = -2, z = 0$

17. $f(x) = x^3 - 4x^2;\quad x = 0, x = 10$

18. $g(t) = t^4 - t^3 + t^2;\quad t = -2, t = 2$

19. $f(t) = 5t^2;\quad t = 3, t = 3 + h$

20. $f(x) = 1 - 3x^2;\quad x = 2, x = 2 + h$

21. $g(x) = \dfrac{1}{x};\quad x = 1, x = a$

22. $g(x) = \dfrac{2}{x + 1};\quad x = 0, x = h$

23. $f(t) = \dfrac{2}{t};\quad t = a, t = a + h$

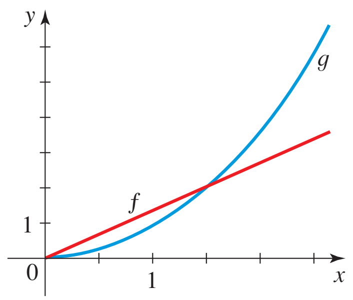
24. $f(t) = \sqrt{t};\quad t = a, t = a + h$

25–26 ■ **Average Rate of Change of a Linear Function** A linear function is given. **(a)** Find the average rate of change of the function between $x = a$ and $x = a + h$. **(b)** Show that the average rate of change is the same as the slope of the line.

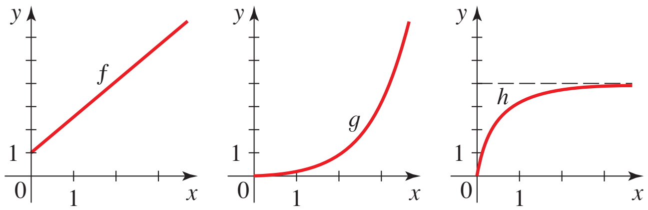
25. $f(x) = \frac{1}{2}x + 3$ **26.** $g(x) = -4x + 2$

SKILLS Plus

27. Average Rate of Change The graphs of the functions f and g are shown. The function ____ (f or g) has a greater average rate of change between $x = 0$ and $x = 1$. The function ____ (f or g) has a greater average rate of change between $x = 1$ and $x = 2$. The functions f and g have the same average rate of change between $x =$ ____ and $x =$ ____.

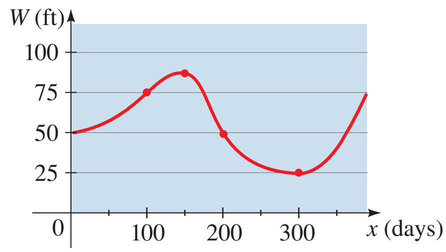


28. Average Rate of Change Graphs of the functions f, g, and h are shown below. What can you say about the average rate of change of each function on the successive intervals $[0, 1], [1, 2], [2, 3], \ldots$?



APPLICATIONS

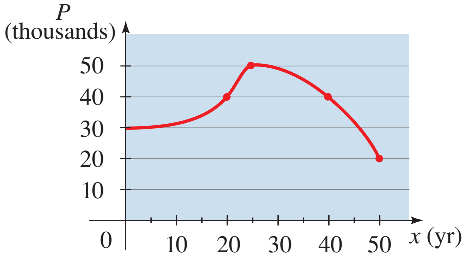
29. Changing Water Levels The graph shows the depth of water W in a reservoir over a one-year period as a function of the number of days x since the beginning of the year. What was the average rate of change of W between $x = 100$ and $x = 200$?



30. Population Growth and Decline The graph shows the population P in a small industrial city from 1950 to 2000. The variable x represents the number of years since 1950.

(a) What was the average rate of change of P between $x = 20$ and $x = 40$?

(b) Interpret the value of the average rate of change that you found in part (a).



31. Population Growth and Decline The table gives the population in a small coastal community for the period 1997–2006. Figures shown are for January 1 in each year.

(a) What was the average rate of change of population between 1998 and 2001?

(b) What was the average rate of change of population between 2002 and 2004?

(c) For what period of time was the population increasing?

(d) For what period of time was the population decreasing?

Year	Population
1997	624
1998	856
1999	1,336
2000	1,578
2001	1,591
2002	1,483
2003	994
2004	826
2005	801
2006	745

32. Running Speed A man is running around a circular track that is 200 m in circumference. An observer uses a stopwatch to record the runner's time at the end of each lap, obtaining the data in the following table.

(a) What was the man's average speed (rate) between 68 s and 152 s?

(b) What was the man's average speed between 263 s and 412 s?

(c) Calculate the man's speed for each lap. Is he slowing down, speeding up, or neither?

Time (s)	Distance (m)
32	200
68	400
108	600
152	800
203	1000
263	1200
335	1400
412	1600

33. DVD Player Sales The table shows the number of DVD players sold in a small electronics store in the years 2003–2013.

Year	DVD players sold
2003	495
2004	513
2005	410
2006	402
2007	520
2008	580
2009	631
2010	719
2011	624
2012	582
2013	635

(a) What was the average rate of change of sales between 2003 and 2013?

(b) What was the average rate of change of sales between 2003 and 2004?

(c) What was the average rate of change of sales between 2004 and 2005?

(d) Between which two successive years did DVD player sales *increase* most quickly? *Decrease* most quickly?

34. Book Collection Between 1980 and 2000 a rare book collector purchased books for his collection at the rate of 40 books per year. Use this information to complete the following table. (Note that not every year is given in the table.)

Year	Number of books	Year	Number of books
1980	420	1995	
1981	460	1997	
1982		1998	
1985		1999	
1990		2000	1220
1992			

35. Cooling Soup When a bowl of hot soup is left in a room, the soup eventually cools down to room temperature. The temperature T of the soup is a function of time t. The table below gives the temperature (in °F) of a bowl of soup t minutes after it was set on the table. Find the average rate of change of the temperature of the soup over the first 20 minutes and over the next 20 minutes. During which interval did the soup cool off more quickly?

t (min)	T (°F)	t (min)	T (°F)
0	200	35	94
5	172	40	89
10	150	50	81
15	133	60	77
20	119	90	72
25	108	120	70
30	100	150	70