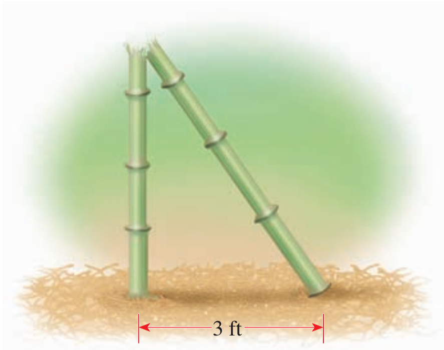
stem, as shown in the figure. What is the height of the break?

[*Hint:* Use the Pythagorean Theorem.]




## DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

**92. WRITE: Historical Research** Read the biographical notes on Pythagoras (page 241), Euclid (page 542), and Archimedes (page 787). Choose one of these mathematicians, and find out more about him from the library or on the Internet. Write a short essay on your findings. Include both biographical information and a description of the mathematics for which he is famous.

**93. WRITE: Real-world Equations** In this section we learned how to translate words into algebra. In this exercise we try to find real-world situations that could correspond to an algebraic equation. For instance, the equation $A = (x + y)/2$ could model the average amount of money in two bank accounts, where $x$ represents the amount in one account and $y$ the amount in the other. Write a story that could correspond to the given equation, stating what the variables represent.

**(a)** $C = 20{,}000 + 4.50x$

**(b)** $A = w(w + 10)$

**(c)** $C = 10.50x + 11.75y$

**94. DISCUSS: A Babylonian Quadratic Equation** The ancient Babylonians knew how to solve quadratic equations. Here is a problem from a cuneiform tablet found in a Babylonian school dating back to about 2000 B.C.

> I have a reed, I know not its length. I broke from it one cubit, and it fit 60 times along the length of my field. I restored to the reed what I had broken off, and it fit 30 times along the width of my field. The area of my field is 375 square nindas. What was the original length of the reed?

Solve this problem. Use the fact that 1 ninda = 12 cubits.

# 1.8 INEQUALITIES

■ Solving Linear Inequalities ■ Solving Nonlinear Inequalities ■ Absolute Value Inequalities ■ Modeling with Inequalities

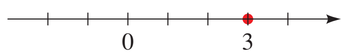
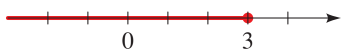
Some problems in algebra lead to **inequalities** instead of equations. An inequality looks just like an equation, except that in the place of the equal sign is one of the symbols, $<$, $>$, $\le$, or $\ge$. Here is an example of an inequality:

$$4x + 7 \le 19$$

| $x$ | $4x + 7 \le 19$ |
|---|---|
| 1 | $11 \le 19$ ✓ |
| 2 | $15 \le 19$ ✓ |
| 3 | $19 \le 19$ ✓ |
| 4 | $23 \le 19$ ✗ |
| 5 | $27 \le 19$ ✗ |

The table in the margin shows that some numbers satisfy the inequality and some numbers don't.

To **solve** an inequality that contains a variable means to find all values of the variable that make the inequality true. Unlike an equation, an inequality generally has infinitely many solutions, which form an interval or a union of intervals on the real line. The following illustration shows how an inequality differs from its corresponding equation:

| | | Solution | Graph |
|---|---|---|---|
| Equation: | $4x + 7 = 19$ | $x = 3$ |  |
| Inequality: | $4x + 7 \le 19$ | $x \le 3$ |  |

To solve inequalities, we use the following rules to isolate the variable on one side of the inequality sign. These rules tell us when two inequalities are *equivalent* (the symbol ⇔ means "is equivalent to"). In these rules the symbols $A$, $B$, and $C$ stand for real numbers or algebraic expressions. Here we state the rules for inequalities involving the symbol $\le$, but they apply to all four inequality symbols.

### RULES FOR INEQUALITIES

| Rule | Description |
|---|---|
| **1.** $A \le B \iff A + C \le B + C$ | **Adding** the same quantity to each side of an inequality gives an equivalent inequality. |
| **2.** $A \le B \iff A - C \le B - C$ | **Subtracting** the same quantity from each side of an inequality gives an equivalent inequality. |
| **3.** If $C > 0$, then $A \le B \iff CA \le CB$ | **Multiplying** each side of an inequality by the same *positive* quantity gives an equivalent inequality. |
| **4.** If $C < 0$, then $A \le B \iff CA \ge CB$ | **Multiplying** each side of an inequality by the same *negative* quantity *reverses the direction* of the inequality. |
| **5.** If $A > 0$ and $B > 0$, then $A \le B \iff \frac{1}{A} \ge \frac{1}{B}$ | **Taking reciprocals** of each side of an inequality involving *positive* quantities *reverses the direction* of the inequality. |
| **6.** If $A \le B$ and $C \le D$, then $A + C \le B + D$ | Inequalities can be added. |
| **7.** If $A \le B$ and $B \le C$, then $A \le C$ | Inequality is transitive. |

Pay special attention to Rules 3 and 4. Rule 3 says that we can multiply (or divide) each side of an inequality by a *positive* number, but Rule 4 says that if we multiply each side of an inequality by a *negative* number, then we reverse the direction of the inequality. For example, if we start with the inequality

$$3 < 5$$

and multiply by 2, we get

$$6 < 10$$

but if we multiply by $-2$, we get

$$-6 > -10$$

## ■ Solving Linear Inequalities

An inequality is **linear** if each term is constant or a multiple of the variable. To solve a linear inequality, we isolate the variable on one side of the inequality sign.

### EXAMPLE 1 ■ Solving a Linear Inequality

Solve the inequality $3x < 9x + 4$, and sketch the solution set.

**SOLUTION**

| | |
|---|---|
| $3x < 9x + 4$ | Given inequality |
| $3x - 9x < 9x + 4 - 9x$ | Subtract $9x$ |
| $-6x < 4$ | Simplify |
| $\left(-\frac{1}{6}\right)(-6x) > \left(-\frac{1}{6}\right)(4)$ | Multiply by $-\frac{1}{6}$ and reverse inequality |
| $x > -\frac{2}{3}$ | Simplify |

Multiplying by the negative number $-\frac{1}{6}$ *reverses* the direction of the inequality.

The solution set consists of all numbers greater than $-\frac{2}{3}$. In other words the solution of the inequality is the interval $\left(-\frac{2}{3}, \infty\right)$. It is graphed in Figure 1.


**FIGURE 1**

Now Try Exercise 21

### EXAMPLE 2 ■ Solving a Pair of Simultaneous Inequalities

Solve the inequalities $4 \le 3x - 2 < 13$.

**SOLUTION** The solution set consists of all values of $x$ that satisfy both of the inequalities $4 \le 3x - 2$ and $3x - 2 < 13$. Using Rules 1 and 3, we see that the following inequalities are equivalent:

$$4 \le 3x - 2 < 13 \qquad \text{Given inequality}$$

$$6 \le 3x < 15 \qquad \text{Add 2}$$

$$2 \le x < 5 \qquad \text{Divide by 3}$$

FIGURE 2

Therefore the solution set is $[2, 5)$, as shown in Figure 2.

Now Try Exercise 33 ■

## ■ Solving Nonlinear Inequalities

To solve inequalities involving squares and other powers of the variable, we use factoring, together with the following principle.

### THE SIGN OF A PRODUCT OR QUOTIENT

- If a product or a quotient has an *even* number of *negative* factors, then its value is *positive*.
- If a product or a quotient has an *odd* number of *negative* factors, then its value is *negative*.

For example, to solve the inequality $x^2 - 5x \le -6$, we first move all terms to the left-hand side and factor to get

$$(x - 2)(x - 3) \le 0$$

This form of the inequality says that the product $(x - 2)(x - 3)$ must be negative or zero, so to solve the inequality, we must determine where each factor is negative or positive (because the sign of a product depends on the sign of the factors). The details are explained in Example 3, in which we use the following guidelines.

### GUIDELINES FOR SOLVING NONLINEAR INEQUALITIES

1. **Move All Terms to One Side.** If necessary, rewrite the inequality so that all nonzero terms appear on one side of the inequality sign. If the nonzero side of the inequality involves quotients, bring them to a common denominator.
2. **Factor.** Factor the nonzero side of the inequality.
3. **Find the Intervals.** Determine the values for which each factor is zero. These numbers will divide the real line into intervals. List the intervals that are determined by these numbers.
4. **Make a Table or Diagram.** Use **test values** to make a table or diagram of the signs of each factor on each interval. In the last row of the table determine the sign of the product (or quotient) of these factors.
5. **Solve.** Use the sign table to find the intervals on which the inequality is satisfied. Check whether the **endpoints** of these intervals satisfy the inequality. (This may happen if the inequality involves $\le$ or $\ge$.)

The factoring technique that is described in these guidelines works only if all nonzero terms appear on one side of the inequality symbol. If the inequality is not written in this form, first rewrite it, as indicated in Step 1.

## EXAMPLE 3 ■ Solving a Quadratic Inequality

Solve the inequality $x^2 \le 5x - 6$.

**SOLUTION** We will follow the guidelines given above.

**Move all terms to one side.** We move all the terms to the left-hand side.

$$x^2 \le 5x - 6 \qquad \text{Given inequality}$$

$$x^2 - 5x + 6 \le 0 \qquad \text{Subtract } 5x\text{, add } 6$$

**Factor.** Factoring the left-hand side of the inequality, we get

$$(x - 2)(x - 3) \le 0 \qquad \text{Factor}$$

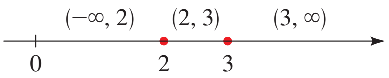



**FIGURE 3**

**Find the intervals.** The factors of the left-hand side are $x - 2$ and $x - 3$. These factors are zero when $x$ is 2 and 3, respectively. As shown in Figure 3, the numbers 2 and 3 divide the real line into the three intervals

$$(-\infty, 2), (2, 3), (3, \infty)$$

The factors $x - 2$ and $x - 3$ change sign only at 2 and 3, respectively. So these factors maintain their sign on each of these three intervals.

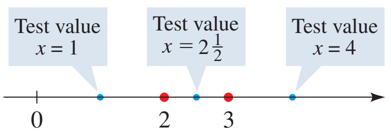



**FIGURE 4**

**Make a table or diagram.** To determine the sign of each factor on each of the intervals that we found, we use test values. We choose a number inside each interval and check the sign of the factors $x - 2$ and $x - 3$ at the number we chose. For the interval $(-\infty, 2)$, let's choose the test value 1 (see Figure 4). Substituting 1 for $x$ in the factors $x - 2$ and $x - 3$, we get

$$x - 2 = 1 - 2 = -1 < 0$$

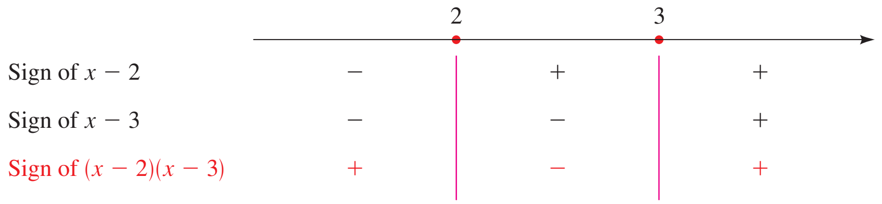
$$x - 3 = 1 - 3 = -2 < 0$$

So both factors are negative on this interval. Notice that we need to check only one test value for each interval because the factors $x - 2$ and $x - 3$ do not change sign on any of the three intervals we found.

Using the test values $x = 2\frac{1}{2}$ and $x = 4$ for the intervals $(2, 3)$ and $(3, \infty)$ (see Figure 4), respectively, we construct the following sign table. The final row of the table is obtained from the fact that the expression in the last row is the product of the two factors.

| Interval | $(-\infty, 2)$ | $(2, 3)$ | $(3, \infty)$ |
|---|---|---|---|
| Sign of $x - 2$ | − | + | + |
| Sign of $x - 3$ | − | − | + |
| Sign of $(x - 2)(x - 3)$ | + | − | + |

If you prefer, you can represent this information on a real line, as in the following sign diagram. The vertical lines indicate the points at which the real line is divided into intervals:



**Solve.** We read from the table or the diagram that $(x-2)(x-3)$ is negative on the interval $(2, 3)$. You can check that the endpoints 2 and 3 satisfy the inequality, so the solution is

$$\{x \mid 2 \le x \le 3\} = [2, 3]$$

The solution is illustrated in Figure 5.

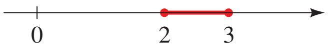



FIGURE 5

Now Try Exercise 43 ■

## EXAMPLE 4 ■ Solving an Inequality with Repeated Factors

Solve the inequality $x(x-1)^2(x-3) < 0$.

**SOLUTION** All nonzero terms are already on one side of the inequality, and the nonzero side of the inequality is already factored. So we begin by finding the intervals for this inequality.

**Find the intervals.** The factors of the left-hand side are $x$, $(x-1)^2$, and $x-3$. These are zero when $x = 0, 1, 3$. These numbers divide the real line into the intervals

$$(-\infty, 0), (0, 1), (1, 3), (3, \infty)$$

**Make a diagram.** We make the following diagram, using test points to determine the sign of each factor in each interval.

| | | 0 | | 1 | | 3 | |
|---|---|---|---|---|---|---|---|
| Sign of $x$ | $-$ | | $+$ | | $+$ | | $+$ |
| Sign of $(x-1)^2$ | $+$ | | $+$ | | $+$ | | $+$ |
| Sign of $(x-3)$ | $-$ | | $-$ | | $-$ | | $+$ |
| Sign of $x(x-1)^2(x-3)$ | $+$ | | $-$ | | $-$ | | $+$ |

**Solve.** From the diagram we see that the inequality is satisfied on the intervals $(0, 1)$ and $(1, 3)$. Since this inequality involves $<$, the endpoints of the intervals do not satisfy the inequality. So the solution set is the union of these two intervals:

$$(0, 1) \cup (1, 3)$$

The solution set is graphed in Figure 6.

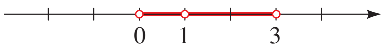



FIGURE 6

Now Try Exercise 55 ■

## EXAMPLE 5 ■ Solving an Inequality Involving a Quotient

Solve the inequality $\dfrac{1+x}{1-x} \ge 1$.

**SOLUTION** **Move all terms to one side.** We move the terms to the left-hand side and simplify using a common denominator.

$$\frac{1+x}{1-x} \ge 1 \qquad \text{Given inequality}$$

$$\frac{1+x}{1-x} - 1 \ge 0 \qquad \text{Subtract 1}$$

$$\frac{1+x}{1-x} - \frac{1-x}{1-x} \ge 0 \qquad \text{Common denominator } 1-x$$

$$\frac{1+x-1+x}{1-x} \ge 0 \qquad \text{Combine the fractions}$$

$$\frac{2x}{1-x} \ge 0 \qquad \text{Simplify}$$

It is tempting to simply multiply both sides of the inequality by $1-x$ (as you would if this were an *equation*). But this doesn't work because we don't know whether $1-x$ is positive or negative, so we can't tell whether the inequality needs to be reversed. (See Exercise 127.)

**Find the intervals.** The factors of the left-hand side are $2x$ and $1 - x$. These are zero when $x$ is 0 and 1. These numbers divide the real line into the intervals

$$(-\infty, 0),\ (0, 1),\ (1, \infty)$$

**Make a diagram.** We make the following diagram using test points to determine the sign of each factor in each interval.

| | | 0 | | 1 | |
|---|---|---|---|---|---|
| Sign of $2x$ | $-$ | | $+$ | | $+$ |
| Sign of $1 - x$ | $+$ | | $+$ | | $-$ |
| Sign of $\dfrac{2x}{1-x}$ | $-$ | | $+$ | | $-$ |

**Solve.** From the diagram we see that the inequality is satisfied on the interval $(0, 1)$. Checking the endpoints, we see that 0 satisfies the inequality but 1 does not (because the quotient in the inequality is not defined at 1). So the solution set is the interval

$$[0, 1)$$

The solution set is graphed in Figure 7.


**FIGURE 7**

Now Try Exercise 61 ■

Example 5 shows that we should always check the endpoints of the solution set to see whether they satisfy the original inequality.

## ■ Absolute Value Inequalities

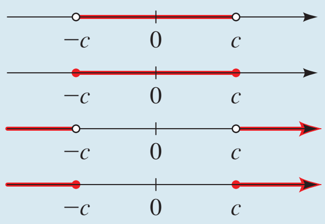
We use the following properties to solve inequalities that involve absolute value.

These properties hold when $x$ is replaced by any algebraic expression. (In the graphs we assume that $c > 0$.)

### PROPERTIES OF ABSOLUTE VALUE INEQUALITIES

| Inequality | Equivalent form |
|---|---|
| **1.** $\lvert x \rvert < c$ | $-c < x < c$ |
| **2.** $\lvert x \rvert \le c$ | $-c \le x \le c$ |
| **3.** $\lvert x \rvert > c$ | $x < -c$ or $c < x$ |
| **4.** $\lvert x \rvert \ge c$ | $x \le -c$ or $c \le x$ |

**Graph**




These properties can be proved using the definition of absolute value. To prove Property 1, for example, note that the inequality $\lvert x \rvert < c$ says that the distance from $x$ to 0 is less than $c$, and from Figure 8 you can see that this is true if and only if $x$ is between $-c$ and $c$.

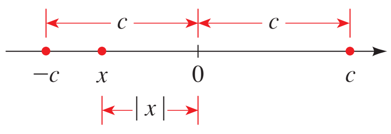



**FIGURE 8**

## EXAMPLE 6 ■ Solving an Absolute Value Inequality

Solve the inequality $\lvert x - 5 \rvert < 2$.

**SOLUTION 1** The inequality $\lvert x - 5 \rvert < 2$ is equivalent to

$$-2 < x - 5 < 2 \qquad \text{Property 1}$$

$$3 < x < 7 \qquad \text{Add 5}$$

The solution set is the open interval $(3, 7)$.

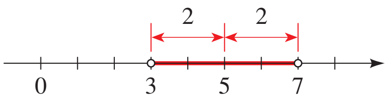



FIGURE 9

**SOLUTION 2** Geometrically, the solution set consists of all numbers $x$ whose distance from 5 is less than 2. From Figure 9 we see that this is the interval $(3, 7)$.

Now Try Exercise 81 ■

### EXAMPLE 7 ■ Solving an Absolute Value Inequality

Solve the inequality $|3x + 2| \geq 4$.

**SOLUTION** By Property 4 the inequality $|3x + 2| \geq 4$ is equivalent to

$$3x + 2 \geq 4 \quad \text{or} \quad 3x + 2 \leq -4$$

$$3x \geq 2 \qquad 3x \leq -6 \qquad \text{Subtract 2}$$

$$x \geq \tfrac{2}{3} \qquad x \leq -2 \qquad \text{Divide by 3}$$

So the solution set is

$$\{x \mid x \leq -2 \quad \text{or} \quad x \geq \tfrac{2}{3}\} = (-\infty, -2] \cup [\tfrac{2}{3}, \infty)$$

The set is graphed in Figure 10.

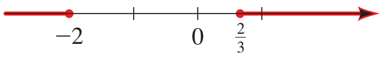



FIGURE 10

Now Try Exercise 83 ■

## ■ Modeling with Inequalities

Modeling real-life problems frequently leads to inequalities because we are often interested in determining when one quantity is more (or less) than another.

### EXAMPLE 8 ■ Carnival Tickets

A carnival has two plans for tickets.

Plan A: $5 entrance fee and 25¢ each ride

Plan B: $2 entrance fee and 50¢ each ride

How many rides would you have to take for Plan A to be less expensive than Plan B?

**SOLUTION** **Identify the variable.** We are asked for the number of rides for which Plan A is less expensive than Plan B. So let

$$x = \text{number of rides}$$

**Translate from words to algebra.** The information in the problem may be organized as follows.

| In Words | In Algebra |
|---|---|
| Number of rides | $x$ |
| Cost with Plan A | $5 + 0.25x$ |
| Cost with Plan B | $2 + 0.50x$ |

**Set up the model.** Now we set up the model.

$$\boxed{\text{cost with Plan A}} < \boxed{\text{cost with Plan B}}$$

$$5 + 0.25x < 2 + 0.50x$$

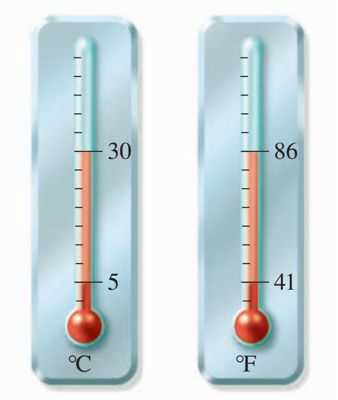
**Solve.** Now we solve for $x$.

$$3 + 0.25x < 0.50x \qquad \text{Subtract 2}$$
$$3 < 0.25x \qquad \text{Subtract } 0.25x$$
$$12 < x \qquad \text{Divide by 0.25}$$

So if you plan to take *more than* 12 rides, Plan A is less expensive.

Now Try Exercise 111 ■

## EXAMPLE 9 ■ Relationship Between Fahrenheit and Celsius Scales




The instructions on a bottle of medicine indicate that the bottle should be stored at a temperature between 5 °C and 30 °C. What range of temperatures does this correspond to on the Fahrenheit scale?

**SOLUTION** The relationship between degrees Celsius ($C$) and degrees Fahrenheit ($F$) is given by the equation $C = \frac{5}{9}(F - 32)$. Expressing the statement on the bottle in terms of inequalities, we have

$$5 < C < 30$$

So the corresponding Fahrenheit temperatures satisfy the inequalities

$$5 < \tfrac{5}{9}(F - 32) < 30 \qquad \text{Substitute } C = \tfrac{5}{9}(F - 32)$$
$$\tfrac{9}{5} \cdot 5 < F - 32 < \tfrac{9}{5} \cdot 30 \qquad \text{Multiply by } \tfrac{9}{5}$$
$$9 < F - 32 < 54 \qquad \text{Simplify}$$
$$9 + 32 < F < 54 + 32 \qquad \text{Add 32}$$
$$41 < F < 86 \qquad \text{Simplify}$$

The medicine should be stored at a temperature between 41°F and 86°F.

Now Try Exercise 109 ■

## 1.8 EXERCISES

### CONCEPTS

**1.** Fill in the blank with an appropriate inequality sign.

**(a)** If $x < 5$, then $x - 3$ _____ 2.

**(b)** If $x \le 5$, then $3x$ _____ 15.

**(c)** If $x \ge 2$, then $-3x$ _____ $-6$.

**(d)** If $x < -2$, then $-x$ _____ 2.

**2.** To solve the nonlinear inequality $\frac{x+1}{x-2} \le 0$, we first observe that the numbers ____ and ____ are zeros of the numerator and denominator. These numbers divide the real line into three intervals. Complete the table.

| **Interval** | | | |
|---|---|---|---|
| **Sign of $x + 1$**<br>**Sign of $x - 2$** | | | |
| **Sign of $(x + 1)/(x - 2)$** | | | |

Do any of the endpoints fail to satisfy the inequality? If so, which one(s)? ________. The solution of the inequality is ______________.

**3. (a)** The solution of the inequality $|x| \le 3$ is the interval ________.

**(b)** The solution of the inequality $|x| \ge 3$ is a union of two intervals ________ ∪ ________.

**4. (a)** The set of all points on the real line whose distance from zero is less than 3 can be described by the absolute value inequality $|x|$ ________.

**(b)** The set of all points on the real line whose distance from zero is greater than 3 can be described by the absolute value inequality $|x|$ ________.

**5.** *Yes or No*? If *No*, give an example.

**(a)** If $x(x + 1) > 0$, does it follow that $x$ is positive?

**(b)** If $x(x + 1) > 5$, does it follow that $x > 5$?

**6.** What is a logical first step in solving the inequality?

**(a)** $3x \le 7$ **(b)** $5x - 2 \ge 1$ **(c)** $|3x + 2| \le 8$

## SKILLS

**7–12 ■ Solutions?** Let $S = \{-5, -1, 0, \frac{2}{3}, \frac{5}{6}, 1, \sqrt{5}, 3, 5\}$. Determine which elements of $S$ satisfy the inequality.

**7.** $-2 + 3x \geq \frac{1}{3}$

**8.** $1 - 2x \geq 5x$

**9.** $1 < 2x - 4 \leq 7$

**10.** $-2 \leq 3 - x < 2$

**11.** $\dfrac{1}{x} \leq \dfrac{1}{2}$

**12.** $x^2 + 2 < 4$

**13–36 ■ Linear Inequalities** Solve the linear inequality. Express the solution using interval notation and graph the solution set.

**13.** $2x \leq 7$

**14.** $-4x \geq 10$

**15.** $2x - 5 > 3$

**16.** $3x + 11 < 5$

**17.** $7 - x \geq 5$

**18.** $5 - 3x \leq -16$

**19.** $2x + 1 < 0$

**20.** $0 < 5 - 2x$

**21.** $4x - 7 < 8 + 9x$

**22.** $5 - 3x \geq 8x - 7$

**23.** $\frac{1}{2}x - \frac{2}{3} > 2$

**24.** $\frac{2}{5}x + 1 < \frac{1}{5} - 2x$

**25.** $\frac{1}{3}x + 2 < \frac{1}{6}x - 1$

**26.** $\frac{2}{3} - \frac{1}{2}x \geq \frac{1}{6} + x$

**27.** $4 - 3x \leq -(1 + 8x)$

**28.** $2(7x - 3) \leq 12x + 16$

**29.** $2 \leq x + 5 < 4$

**30.** $5 \leq 3x - 4 \leq 14$

**31.** $-1 < 2x - 5 < 7$

**32.** $1 < 3x + 4 \leq 16$

**33.** $-2 < 8 - 2x \leq -1$

**34.** $-3 \leq 3x + 7 \leq \frac{1}{2}$

**35.** $\dfrac{1}{6} < \dfrac{2x - 13}{12} \leq \dfrac{2}{3}$

**36.** $-\dfrac{1}{2} \leq \dfrac{4 - 3x}{5} \leq \dfrac{1}{4}$

**37–58 ■ Nonlinear Inequalities** Solve the nonlinear inequality. Express the solution using interval notation and graph the solution set.

**37.** $(x + 2)(x - 3) < 0$

**38.** $(x - 5)(x + 4) \geq 0$

**39.** $x(2x + 7) \geq 0$

**40.** $x(2 - 3x) \leq 0$

**41.** $x^2 - 3x - 18 \leq 0$

**42.** $x^2 + 5x + 6 > 0$

**43.** $2x^2 + x \geq 1$

**44.** $x^2 < x + 2$

**45.** $3x^2 - 3x < 2x^2 + 4$

**46.** $5x^2 + 3x \geq 3x^2 + 2$

**47.** $x^2 > 3(x + 6)$

**48.** $x^2 + 2x > 3$

**49.** $x^2 < 4$

**50.** $x^2 \geq 9$

**51.** $(x + 2)(x - 1)(x - 3) \leq 0$

**52.** $(x - 5)(x - 2)(x + 1) > 0$

**53.** $(x - 4)(x + 2)^2 < 0$

**54.** $(x + 3)^2(x + 1) > 0$

**55.** $(x + 3)^2(x - 2)(x + 5) \geq 0$

**56.** $4x^2(x^2 - 9) \leq 0$

**57.** $x^3 - 4x > 0$

**58.** $16x \leq x^3$

**59–74 ■ Inequalities Involving Quotients** Solve the nonlinear inequality. Express the solution using interval notation, and graph the solution set.

**59.** $\dfrac{x - 3}{x + 1} \geq 0$

**60.** $\dfrac{2x + 6}{x - 2} < 0$

**61.** $\dfrac{x}{x + 1} > 3$

**62.** $\dfrac{x - 4}{2x + 1} < 5$

**63.** $\dfrac{2x + 1}{x - 5} \leq 3$

**64.** $\dfrac{3 + x}{3 - x} \geq 1$

**65.** $\dfrac{4}{x} < x$

**66.** $\dfrac{x}{x + 1} > 3x$

**67.** $1 + \dfrac{2}{x + 1} \leq \dfrac{2}{x}$

**68.** $\dfrac{3}{x - 1} - \dfrac{4}{x} \geq 1$

**69.** $\dfrac{6}{x - 1} - \dfrac{6}{x} \geq 1$

**70.** $\dfrac{x}{2} \geq \dfrac{5}{x + 1} + 4$

**71.** $\dfrac{x + 2}{x + 3} < \dfrac{x - 1}{x - 2}$

**72.** $\dfrac{1}{x + 1} + \dfrac{1}{x + 2} \leq 0$

**73.** $x^4 > x^2$

**74.** $x^5 > x^2$

**75–90 ■ Absolute Value Inequalities** Solve the absolute value inequality. Express the answer using interval notation and graph the solution set.

**75.** $|5x| < 20$

**76.** $|16x| \leq 8$

**77.** $|2x| > 7$

**78.** $\frac{1}{2}|x| \geq 1$

**79.** $|x - 5| \leq 3$

**80.** $|x + 1| \geq 1$

**81.** $|3x + 2| < 4$

**82.** $|5x - 2| < 8$

**83.** $|3x - 2| \geq 5$

**84.** $|8x + 3| > 12$

**85.** $\left|\dfrac{x - 2}{3}\right| < 2$

**86.** $\left|\dfrac{x + 1}{2}\right| \geq 4$

**87.** $|x + 6| < 0.001$

**88.** $3 - |2x + 4| \leq 1$

**89.** $8 - |2x - 1| \geq 6$

**90.** $7|x + 2| + 5 > 4$

**91–94 ■ Absolute Value Inequalities** A phrase describing a set of real numbers is given. Express the phrase as an inequality involving an absolute value.

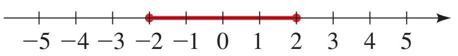
**91.** All real numbers $x$ less than 3 units from 0

**92.** All real numbers $x$ more than 2 units from 0

**93.** All real numbers $x$ at least 5 units from 7

**94.** All real numbers $x$ at most 4 units from 2

**95–100 ■ Absolute Value Inequalities** A set of real numbers is graphed. Find an inequality involving an absolute value that describes the set.

**95.**
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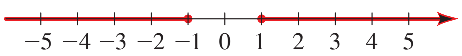


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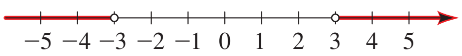


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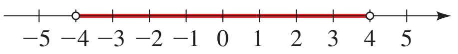


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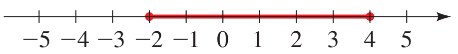
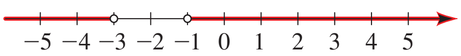



**100.**



**101–104 ■ Domain** Determine the values of the variable for which the expression is defined as a real number.

**101.** $\sqrt{x^2 - 9}$

**102.** $\sqrt{x^2 - 5x - 50}$

**103.** $\left(\dfrac{1}{x^2 - 3x - 10}\right)^{1/2}$ **104.** $\sqrt[4]{\dfrac{1 - x}{2 + x}}$

## SKILLS Plus

**105–108 ■ Inequalities** Solve the inequality for $x$. Assume that $a$, $b$, and $c$ are positive constants.

**105.** $a(bx - c) \geq bc$ **106.** $a \leq bx + c < 2a$

**107.** $a|bx - c| + d \geq 4a$ **108.** $\left|\dfrac{bx + c}{a}\right| > 5a$

## APPLICATIONS

**109. Temperature Scales** Use the relationship between $C$ and $F$ given in Example 9 to find the interval on the Fahrenheit scale corresponding to the temperature range $20 \leq C \leq 30$.

**110. Temperature Scales** What interval on the Celsius scale corresponds to the temperature range $50 \leq F \leq 95$?

**111. Car Rental Cost** A car rental company offers two plans for renting a car.

Plan A: \$30 per day and 10¢ per mile
Plan B: \$50 per day with free unlimited mileage

For what range of miles will Plan B save you money?

**112. International Plans** A phone service provider offers two international plans.

Plan A: \$25 per month and 5¢ per minute
Plan B: \$5 per month and 12¢ per minute

For what range of minutes of international calls would Plan B be financially advantageous?

**113. Driving Cost** It is estimated that the annual cost of driving a certain new car is given by the formula

$$C = 0.35m + 2200$$

where $m$ represents the number of miles driven per year and $C$ is the cost in dollars. Jane has purchased such a car and decides to budget between \$6400 and \$7100 for next year's driving costs. What is the corresponding range of miles that she can drive her new car?

**114. Air Temperature** As dry air moves upward, it expands and, in so doing, cools at a rate of about 1°C for each 100-m rise, up to about 12 km.

**(a)** If the ground temperature is 20°C, write a formula for the temperature at height $h$.

**(b)** What range of temperatures can be expected if a plane takes off and reaches a maximum height of 5 km?

**115. Airline Ticket Price** A charter airline finds that on its Saturday flights from Philadelphia to London all 120 seats will be sold if the ticket price is \$200. However, for each \$3 increase in ticket price, the number of seats sold decreases by one.

**(a)** Find a formula for the number of seats sold if the ticket price is $P$ dollars.

**(b)** Over a certain period the number of seats sold for this flight ranged between 90 and 115. What was the corresponding range of ticket prices?

**116. Accuracy of a Scale** A coffee merchant sells a customer 3 lb of Hawaiian Kona at \$6.50 per pound. The merchant's scale is accurate to within ±0.03 lb. By how much could the customer have been overcharged or undercharged because of possible inaccuracy in the scale?

**117. Gravity** The gravitational force $F$ exerted by the earth on an object having a mass of 100 kg is given by the equation

$$F = \frac{4{,}000{,}000}{d^2}$$

where $d$ is the distance (in km) of the object from the center of the earth, and the force $F$ is measured in newtons (N). For what distances will the gravitational force exerted by the earth on this object be between 0.0004 N and 0.01 N?

**118. Bonfire Temperature** In the vicinity of a bonfire the temperature $T$ in °C at a distance of $x$ meters from the center of the fire was given by

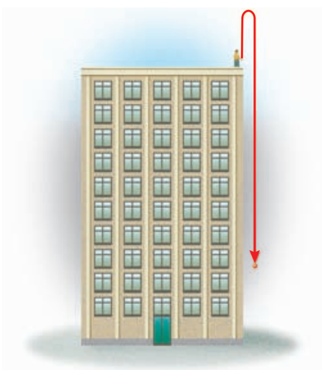
$$T = \frac{600{,}000}{x^2 + 300}$$

At what range of distances from the fire's center was the temperature less than 500°C?

**119. Falling Ball** Using calculus, it can be shown that if a ball is thrown upward with an initial velocity of 16 ft/s from the top of a building 128 ft high, then its height $h$ above the ground $t$ seconds later will be

$$h = 128 + 16t - 16t^2$$

During what time interval will the ball be at least 32 ft above the ground?

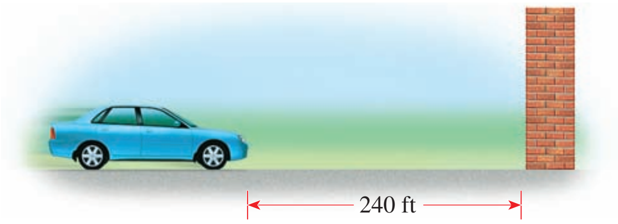


**120. Gas Mileage** The gas mileage $g$ (measured in mi/gal) for a particular vehicle, driven at $v$ mi/h, is given by the formula $g = 10 + 0.9v - 0.01v^2$, as long as $v$ is between 10 mi/h and 75 mi/h. For what range of speeds is the vehicle's mileage 30 mi/gal or better?

**121. Stopping Distance** For a certain model of car the distance $d$ required to stop the vehicle if it is traveling at $v$ mi/h is given by the formula

$$d = v + \frac{v^2}{20}$$

where $d$ is measured in feet. Kerry wants her stopping distance not to exceed 240 ft. At what range of speeds can she travel?




**122. Manufacturer's Profit** If a manufacturer sells $x$ units of a certain product, revenue $R$ and cost $C$ (in dollars) are given by

$$R = 20x$$

$$C = 2000 + 8x + 0.0025x^2$$

Use the fact that

$$\text{profit} = \text{revenue} - \text{cost}$$

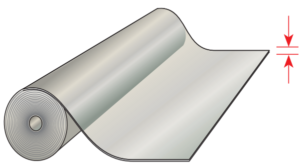
to determine how many units the manufacturer should sell to enjoy a profit of at least \$2400.

**123. Fencing a Garden** A determined gardener has 120 ft of deer-resistant fence. She wants to enclose a rectangular vegetable garden in her backyard, and she wants the area that is enclosed to be at least 800 ft$^2$. What range of values is possible for the length of her garden?

**124. Thickness of a Laminate** A company manufactures industrial laminates (thin nylon-based sheets) of thickness 0.020 in., with a tolerance of 0.003 in.

**(a)** Find an inequality involving absolute values that describes the range of possible thickness for the laminate.

**(b)** Solve the inequality you found in part (a).




**125. Range of Height** The average height of adult males is 68.2 in., and 95% of adult males have height $h$ that satisfies the inequality

$$\left|\frac{h - 68.2}{2.9}\right| \le 2$$

Solve the inequality to find the range of heights.

## DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

**126. DISCUSS ■ DISCOVER: Do Powers Preserve Order?** If $a < b$, is $a^2 < b^2$? (Check both positive and negative values for $a$ and $b$.) If $a < b$, is $a^3 < b^3$? On the basis of your observations, state a general rule about the relationship between $a^n$ and $b^n$ when $a < b$ and $n$ is a positive integer.

**127. DISCUSS ■ DISCOVER: What's Wrong Here?** It is tempting to try to solve an inequality like an equation. For instance, we might try to solve $1 < 3/x$ by multiplying both sides by $x$, to get $x < 3$, so the solution would be $(-\infty, 3)$. But that's wrong; for example, $x = -1$ lies in this interval but does not satisfy the original inequality. Explain why this method doesn't work (think about the *sign* of $x$). Then solve the inequality correctly.

**128. DISCUSS ■ DISCOVER: Using Distances to Solve Absolute Value Inequalities** Recall that $|a - b|$ is the distance between $a$ and $b$ on the number line. For any number $x$, what do $|x - 1|$ and $|x - 3|$ represent? Use this interpretation to solve the inequality $|x - 1| < |x - 3|$ geometrically. In general, if $a < b$, what is the solution of the inequality $|x - a| < |x - b|$?

**129–130 ■ PROVE: Inequalities** Use the properties of inequalities to prove the following inequalities.

**129.** Rule 6 for Inequalities: If $a$, $b$, $c$, and $d$ are any real numbers such that $a < b$ and $c < d$, then $a + c < b + d$. [*Hint:* Use Rule 1 to show that $a + c < b + c$ and $b + c < b + d$. Use Rule 7.]

**130.** If $a$, $b$, $c$, and $d$ are positive numbers such that $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d}$. [*Hint:* Show that $\frac{ad}{b} + a < c + a$ and $a + c < \frac{cb}{d} + c$.]

**131. PROVE: Arithmetic-Geometric Mean Inequality** If $a_1, a_2, \ldots, a_n$ are nonnegative numbers, then their arithmetic mean is $\frac{a_1 + a_2 + \cdots + a_n}{n}$, and their geometric mean is $\sqrt[n]{a_1 a_2 \ldots a_n}$. The arithmetic-geometric mean inequality states that the geometric mean is always less than or equal to the arithmetic mean. In this problem we prove this in the case of two numbers $x$ and $y$.

**(a)** If $x$ and $y$ are nonnegative and $x \le y$, then $x^2 \le y^2$. [*Hint:* First use Rule 3 of Inequalities to show that $x^2 \le xy$ and $xy \le y^2$.]

**(b)** Prove the arithmetic-geometric mean inequality

$$\sqrt{xy} \le \frac{x + y}{2}$$