101. DISCUSS: Algebraic Errors The left-hand column of the table lists some common algebraic errors. In each case, give an example using numbers that shows that the formula is not valid. An example of this type, which shows that a statement is false, is called a *counterexample*.

Algebraic errors	Counterexample
$\frac{1}{a}+\frac{1}{b} \cancel{=} \frac{1}{a+b}$	$\frac{1}{2}+\frac{1}{2} \neq \frac{1}{2+2}$
$(a+b)^2 \cancel{=} a^2+b^2$	
$\sqrt{a^2+b^2} \cancel{=} a+b$	
$\frac{a+b}{a} \cancel{=} b$	
$\frac{a}{a+b} \cancel{=} \frac{1}{b}$	
$\frac{a^m}{a^n} \cancel{=} a^{m/n}$	

102. DISCUSS: Algebraic Errors Determine whether the given equation is true for all values of the variables. If not, give a counterexample. (Disregard any value that makes a denominator zero.)

(a) $\frac{5+a}{5}=1+\frac{a}{5}$ **(b)** $\frac{x+1}{y+1}=\frac{x}{y}$

(c) $\frac{x}{x+y}=\frac{1}{1+y}$ **(d)** $2\left(\frac{a}{b}\right)=\frac{2a}{2b}$

(e) $\frac{-a}{b}=-\frac{a}{b}$ **(f)** $\frac{1+x+x^2}{x}=\frac{1}{x}+1+x$

103. DISCOVER ■ PROVE: Values of a Rational Expression Consider the expression

$$x+\frac{1}{x}$$

for $x > 0$.

(a) Fill in the table, and try other values for x. What do you think is the smallest possible value for this expression?

x	1	3	$\frac{1}{2}$	$\frac{9}{10}$	$\frac{99}{100}$	
$x+\frac{1}{x}$						

(b) Prove that for $x > 0$,

$$x+\frac{1}{x} \geq 2$$

[*Hint:* Multiply by x, move terms to one side, and then factor to arrive at a true statement. Note that each step you made is reversible.]

1.5 EQUATIONS

■ Solving Linear Equations ■ Solving Quadratic Equations ■ Other Types of Equations

An equation is a statement that two mathematical expressions are equal. For example,

$$3+5=8$$

is an equation. Most equations that we study in algebra contain variables, which are symbols (usually letters) that stand for numbers. In the equation

$$4x+7=19$$

the letter x is the variable. We think of x as the "unknown" in the equation, and our goal is to find the value of x that makes the equation true. The values of the unknown that make the equation true are called the **solutions** or **roots** of the equation, and the process of finding the solutions is called **solving the equation**.

$x=3$ is a solution of the equation $4x+7=19$, because substituting $x=3$ makes the equation true:

$x=3$

$4(3)+7=19$ ✓

Two equations with exactly the same solutions are called **equivalent equations**. To solve an equation, we try to find a simpler, equivalent equation in which the variable stands alone on one side of the "equal" sign. Here are the properties that we use to solve an equation. (In these properties, A, B, and C stand for any algebraic expressions, and the symbol $\Leftrightarrow$ means "is equivalent to.")

PROPERTIES OF EQUALITY

Property	Description
1. $A = B \quad \Leftrightarrow \quad A + C = B + C$	Adding the same quantity to both sides of an equation gives an equivalent equation.
2. $A = B \quad \Leftrightarrow \quad CA = CB \quad (C \neq 0)$	Multiplying both sides of an equation by the same nonzero quantity gives an equivalent equation.

These properties require that you *perform the same operation on both sides of an equation* when solving it. Thus if we say "*add* -7" when solving an equation, that is just a short way of saying "*add* -7 to each side of the equation."

Solving Linear Equations

The simplest type of equation is a *linear equation*, or first-degree equation, which is an equation in which each term is either a constant or a nonzero multiple of the variable.

LINEAR EQUATIONS

A **linear equation** in one variable is an equation equivalent to one of the form

$$ax + b = 0$$

where a and b are real numbers and x is the variable.

Here are some examples that illustrate the difference between linear and nonlinear equations.

Linear equations	Nonlinear equations	
$4x - 5 = 3$	$x^2 + 2x = 8$	Not linear; contains the square of the variable
$2x = \frac{1}{2}x - 7$	$\sqrt{x} - 6x = 0$	Not linear; contains the square root of the variable
$x - 6 = \dfrac{x}{3}$	$\dfrac{3}{x} - 2x = 1$	Not linear; contains the reciprocal of the variable

EXAMPLE 1 ■ Solving a Linear Equation

Solve the equation $7x - 4 = 3x + 8$.

SOLUTION We solve this by changing it to an equivalent equation with all terms that have the variable x on one side and all constant terms on the other.

$7x - 4 = 3x + 8$	Given equation
$(7x - 4) + 4 = (3x + 8) + 4$	Add 4
$7x = 3x + 12$	Simplify
$7x - 3x = (3x + 12) - 3x$	Subtract $3x$
$4x = 12$	Simplify
$\frac{1}{4} \cdot 4x = \frac{1}{4} \cdot 12$	Multiply by $\frac{1}{4}$
$x = 3$	Simplify

Because it is important to CHECK YOUR ANSWER, we do this in many of our examples. In these checks, LHS stands for "left-hand side" and RHS stands for "right-hand side" of the original equation.

CHECK YOUR ANSWER

$x = 3$:

$x = 3$ $\qquad$ $x = 3$

$$\text{LHS} = 7(3) - 4 = 17 \qquad \text{RHS} = 3(3) + 8 = 17$$

LHS = RHS ✓

Now Try Exercise 17 ■

Many formulas in the sciences involve several variables, and it is often necessary to express one of the variables in terms of the others. In the next example we solve for a variable in Newton's Law of Gravity.

EXAMPLE 2 ■ Solving for One Variable in Terms of Others

This is Newton's Law of Gravity. It gives the gravitational force F between two masses m and M that are a distance r apart. The constant G is the universal gravitational constant.

Solve for the variable M in the equation

$$F = G\frac{mM}{r^2}$$

SOLUTION Although this equation involves more than one variable, we solve it as usual by isolating M on one side and treating the other variables as we would numbers.

$$F = \left(\frac{Gm}{r^2}\right)M \qquad \text{Factor } M \text{ from RHS}$$

$$\left(\frac{r^2}{Gm}\right)F = \left(\frac{r^2}{Gm}\right)\left(\frac{Gm}{r^2}\right)M \qquad \text{Multiply by reciprocal of } \frac{Gm}{r^2}$$

$$\frac{r^2F}{Gm} = M \qquad \text{Simplify}$$

The solution is $M = \dfrac{r^2F}{Gm}$.

Now Try Exercise 31 ■

EXAMPLE 3 ■ Solving for One Variable in Terms of Others

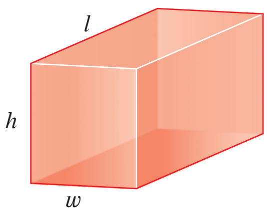


FIGURE 1 A closed rectangular box

The surface area A of the closed rectangular box shown in Figure 1 can be calculated from the length l, the width w, and the height h according to the formula

$$A = 2lw + 2wh + 2lh$$

Solve for w in terms of the other variables in this equation.

SOLUTION Although this equation involves more than one variable, we solve it as usual by isolating w on one side, treating the other variables as we would numbers.

$$A = (2lw + 2wh) + 2lh \qquad \text{Collect terms involving } w$$

$$A - 2lh = 2lw + 2wh \qquad \text{Subtract } 2lh$$

$$A - 2lh = (2l + 2h)w \qquad \text{Factor } w \text{ from RHS}$$

$$\frac{A - 2lh}{2l + 2h} = w \qquad \text{Divide by } 2l + 2h$$

The solution is $w = \dfrac{A - 2lh}{2l + 2h}$.

Now Try Exercise 33 ■

Solving Quadratic Equations

Linear equations are first-degree equations like $2x + 1 = 5$ or $4 - 3x = 2$. Quadratic equations are second-degree equations like $x^2 + 2x - 3 = 0$ or $2x^2 + 3 = 5x$.

Quadratic Equations

$x^2 - 2x - 8 = 0$

$3x + 10 = 4x^2$

$\frac{1}{2}x^2 + \frac{1}{3}x - \frac{1}{6} = 0$

QUADRATIC EQUATIONS

A **quadratic equation** is an equation of the form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers with $a \neq 0$.

Some quadratic equations can be solved by factoring and using the following basic property of real numbers.

ZERO-PRODUCT PROPERTY

$$AB = 0 \quad \text{if and only if} \quad A = 0 \quad \text{or} \quad B = 0$$

This means that if we can factor the left-hand side of a quadratic (or other) equation, then we can solve it by setting each factor equal to 0 in turn. This method works only when the right-hand side of the equation is 0.

EXAMPLE 4 ■ Solving a Quadratic Equation by Factoring

Find all real solutions of the equation $x^2 + 5x = 24$.

SOLUTION We must first rewrite the equation so that the right-hand side is 0.

$$x^2 + 5x = 24$$

$$x^2 + 5x - 24 = 0 \qquad \text{Subtract 24}$$

$$(x - 3)(x + 8) = 0 \qquad \text{Factor}$$

$$x - 3 = 0 \quad \text{or} \quad x + 8 = 0 \qquad \text{Zero-Product Property}$$

$$x = 3 \qquad\qquad x = -8 \qquad \text{Solve}$$

The solutions are $x = 3$ and $x = -8$.

CHECK YOUR ANSWERS

$x = 3$:

$(3)^2 + 5(3) = 9 + 15 = 24$ ✓

$x = -8$:

$(-8)^2 + 5(-8) = 64 - 40 = 24$ ✓

Now Try Exercise 45 ■

Do you see why one side of the equation must be 0 in Example 4? Factoring the equation as $x(x + 5) = 24$ does not help us find the solutions, since 24 can be factored in infinitely many ways, such as $6 \cdot 4$, $\frac{1}{2} \cdot 48$, $\left(-\frac{2}{5}\right) \cdot (-60)$, and so on.

A quadratic equation of the form $x^2 - c = 0$, where c is a positive constant, factors as $(x - \sqrt{c})(x + \sqrt{c}) = 0$, so the solutions are $x = \sqrt{c}$ and $x = -\sqrt{c}$. We often abbreviate this as $x = \pm\sqrt{c}$.

SOLVING A SIMPLE QUADRATIC EQUATION

The solutions of the equation $x^2 = c$ are $x = \sqrt{c}$ and $x = -\sqrt{c}$.

EXAMPLE 5 ■ Solving Simple Quadratics

Find all real solutions of each equation.

(a) $x^2 = 5$ **(b)** $(x - 4)^2 = 5$

SOLUTION

(a) From the principle in the preceding box we get $x = \pm\sqrt{5}$.

(b) We can take the square root of each side of this equation as well.

$$
\begin{aligned}
(x-4)^2 &= 5 \\
x - 4 &= \pm\sqrt{5} && \text{Take the square root} \\
x &= 4 \pm \sqrt{5} && \text{Add 4}
\end{aligned}
$$

The solutions are $x = 4 + \sqrt{5}$ and $x = 4 - \sqrt{5}$.

Now Try Exercises 53 and 55 ■

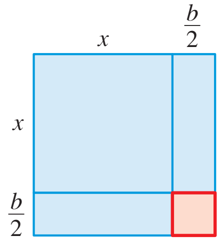
See page 31 for how to recognize when a quadratic expression is a perfect square.

Completing the Square

The area of the blue region is

$$x^2 + 2\left(\frac{b}{2}\right)x = x^2 + bx$$

Add a small square of area $(b/2)^2$ to "complete" the square.



As we saw in Example 5, if a quadratic equation is of the form $(x \pm a)^2 = c$, then we can solve it by taking the square root of each side. In an equation of this form, the left-hand side is a *perfect square*: the square of a linear expression in x. So if a quadratic equation does not factor readily, then we can solve it using the technique of **completing the square**. This means that we add a constant to an expression to make it a perfect square. For example, to make $x^2 - 6x$ a perfect square, we must add 9, since $x^2 - 6x + 9 = (x - 3)^2$.

COMPLETING THE SQUARE

To make $x^2 + bx$ a perfect square, add $\left(\frac{b}{2}\right)^2$, the square of half the coefficient of x. This gives the perfect square

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

EXAMPLE 6 ■ Solving Quadratic Equations by Completing the Square

Find all real solutions of each equation.

(a) $x^2 - 8x + 13 = 0$ **(b)** $3x^2 - 12x + 6 = 0$

SOLUTION

(a)

$$
\begin{aligned}
x^2 - 8x + 13 &= 0 && \text{Given equation} \\
x^2 - 8x &= -13 && \text{Subtract 13} \\
x^2 - 8x + 16 &= -13 + 16 && \text{Complete the square: add } \left(\frac{-8}{2}\right)^2 = 16 \\
(x-4)^2 &= 3 && \text{Perfect square} \\
x - 4 &= \pm\sqrt{3} && \text{Take square root} \\
x &= 4 \pm \sqrt{3} && \text{Add 4}
\end{aligned}
$$

(b) After subtracting 6 from each side of the equation, we must factor the coefficient of x^2 (the 3) from the left side to put the equation in the correct form for completing the square.

$$
\begin{aligned}
3x^2 - 12x + 6 &= 0 && \text{Given equation} \\
3x^2 - 12x &= -6 && \text{Subtract 6} \\
3(x^2 - 4x) &= -6 && \text{Factor 3 from LHS}
\end{aligned}
$$

Now we complete the square by adding $(-2)^2 = 4$ *inside* the parentheses. Since everything inside the parentheses is multiplied by 3, this means that we are

When completing the square, make sure the coefficient of x^2 is 1. If it isn't, you must factor this coefficient from both terms that contain x:

$$ax^2 + bx = a\left(x^2 + \frac{b}{a}x\right)$$

Then complete the square inside the parentheses. Remember that the term added inside the parentheses is multiplied by a.

actually adding $3 \cdot 4 = 12$ to the left side of the equation. Thus we must add 12 to the right side as well.

$$\begin{aligned} 3(x^2 - 4x + 4) &= -6 + 3 \cdot 4 && \text{Complete the square: add 4} \\ 3(x - 2)^2 &= 6 && \text{Perfect square} \\ (x - 2)^2 &= 2 && \text{Divide by 3} \\ x - 2 &= \pm\sqrt{2} && \text{Take square root} \\ x &= 2 \pm \sqrt{2} && \text{Add 2} \end{aligned}$$

Now Try Exercises 57 and 61 ■

We can use the technique of completing the square to derive a formula for the roots of the general quadratic equation $ax^2 + bx + c = 0$.

THE QUADRATIC FORMULA

The roots of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof First, we divide each side of the equation by a and move the constant to the right side, giving

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{Divide by } a$$

We now complete the square by adding $(b/2a)^2$ to each side of the equation:

$$\begin{aligned} x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 && \text{Complete the square: Add } \left(\frac{b}{2a}\right)^2 \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{-4ac + b^2}{4a^2} && \text{Perfect square} \\ x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} && \text{Take square root} \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Subtract } \frac{b}{2a} \end{aligned}$$

■

The Quadratic Formula could be used to solve the equations in Examples 4 and 6. You should carry out the details of these calculations.

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FRANÇOIS VIÈTE (1540–1603) had a successful political career before taking up mathematics late in life. He became one of the most famous French mathematicians of the 16th century. Viète introduced a new level of abstraction in algebra by using letters to stand for *known* quantities in an equation. Before Viète's time, each equation had to be solved on its own. For instance, the quadratic equations

$$3x^2 + 2x + 8 = 0$$

$$5x^2 - 6x + 4 = 0$$

had to be solved separately by completing the square. Viète's idea was to consider all quadratic equations at once by writing

$$ax^2 + bx + c = 0$$

where a, b, and c are known quantities. Thus he made it possible to write a *formula* (in this case the quadratic formula) involving a, b, and c that can be used to solve all such equations in one fell swoop.

Viète's mathematical genius proved quite valuable during a war between France and Spain. To communicate with their troops, the Spaniards used a complicated code that Viète managed to decipher. Unaware of Viète's accomplishment, the Spanish king, Philip II, protested to the Pope, claiming that the French were using witchcraft to read his messages.

EXAMPLE 7 ■ Using the Quadratic Formula

Find all real solutions of each equation.

(a) $3x^2 - 5x - 1 = 0$ **(b)** $4x^2 + 12x + 9 = 0$ **(c)** $x^2 + 2x + 2 = 0$

SOLUTION

(a) In this quadratic equation $a = 3$, $b = -5$, and $c = -1$.

$$\underbrace{3}_{a=3}x^2 \overbrace{-\,5}^{b=-5}x \underbrace{-\,1}_{c=-1} = 0$$

By the Quadratic Formula,

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-1)}}{2(3)} = \frac{5 \pm \sqrt{37}}{6}$$

If approximations are desired, we can use a calculator to obtain

$$x = \frac{5 + \sqrt{37}}{6} \approx 1.8471 \qquad \text{and} \qquad x = \frac{5 - \sqrt{37}}{6} \approx -0.1805$$

Another Method

$$4x^2 + 12x + 9 = 0$$
$$(2x + 3)^2 = 0$$
$$2x + 3 = 0$$
$$x = -\tfrac{3}{2}$$

(b) Using the Quadratic Formula with $a = 4$, $b = 12$, and $c = 9$ gives

$$x = \frac{-12 \pm \sqrt{(12)^2 - 4 \cdot 4 \cdot 9}}{2 \cdot 4} = \frac{-12 \pm 0}{8} = -\frac{3}{2}$$

This equation has only one solution, $x = -\frac{3}{2}$.

(c) Using the Quadratic Formula with $a = 1$, $b = 2$, and $c = 2$ gives

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2\sqrt{-1}}{2} = -1 \pm \sqrt{-1}$$

Since the square of any real number is nonnegative, $\sqrt{-1}$ is undefined in the real number system. The equation has no real solution.

Now Try Exercises 67, 73, and 77 ■

In the next section we study the complex number system, in which the square roots of negative numbers do exist. The equation in Example 7(c) does have solutions in the complex number system.

The quantity $b^2 - 4ac$ that appears under the square root sign in the quadratic formula is called the *discriminant* of the equation $ax^2 + bx + c = 0$ and is given the symbol D. If $D < 0$, then $\sqrt{b^2 - 4ac}$ is undefined, and the quadratic equation has no real solution, as in Example 7(c). If $D = 0$, then the equation has only one real solution, as in Example 7(b). Finally, if $D > 0$, then the equation has two distinct real solutions, as in Example 7(a). The following box summarizes these observations.

THE DISCRIMINANT

The **discriminant** of the general quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ is $D = b^2 - 4ac$.

1. If $D > 0$, then the equation has two distinct real solutions.
2. If $D = 0$, then the equation has exactly one real solution.
3. If $D < 0$, then the equation has no real solution.

EXAMPLE 8 ■ Using the Discriminant

Use the discriminant to determine how many real solutions each equation has.

(a) $x^2 + 4x - 1 = 0$ **(b)** $4x^2 - 12x + 9 = 0$ **(c)** $\frac{1}{3}x^2 - 2x + 4 = 0$

SOLUTION

(a) The discriminant is $D = 4^2 - 4(1)(-1) = 20 > 0$, so the equation has two distinct real solutions.

(b) The discriminant is $D = (-12)^2 - 4 \cdot 4 \cdot 9 = 0$, so the equation has exactly one real solution.

(c) The discriminant is $D = (-2)^2 - 4\left(\frac{1}{3}\right)4 = -\frac{4}{3} < 0$, so the equation has no real solution.

Now Try Exercises 81, 83, and 85 ■

This formula depends on the fact that acceleration due to gravity is constant near the earth's surface. Here we neglect the effect of air resistance.

Now let's consider a real-life situation that can be modeled by a quadratic equation.

EXAMPLE 9 ■ The Path of a Projectile

An object thrown or fired straight upward at an initial speed of v_0 ft/s will reach a height of h feet after t seconds, where h and t are related by the formula

$$h = -16t^2 + v_0 t$$

Suppose that a bullet is shot straight upward with an initial speed of 800 ft/s. Its path is shown in Figure 2.

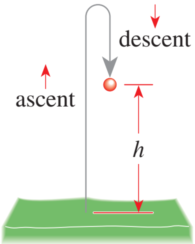


FIGURE 2

(a) When does the bullet fall back to ground level?

(b) When does it reach a height of 6400 ft?

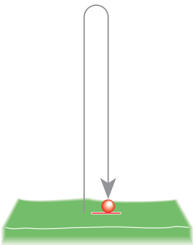
(c) When does it reach a height of 2 mi?

(d) How high is the highest point the bullet reaches?

SOLUTION Since the initial speed in this case is $v_0 = 800$ ft/s, the formula is

$$h = -16t^2 + 800t$$

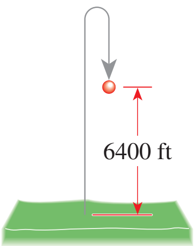
(a) Ground level corresponds to $h = 0$, so we must solve the equation



$$\begin{aligned} 0 &= -16t^2 + 800t && \text{Set } h = 0 \\ 0 &= -16t(t - 50) && \text{Factor} \end{aligned}$$

Thus $t = 0$ or $t = 50$. This means the bullet starts $(t = 0)$ at ground level and returns to ground level after 50 s.

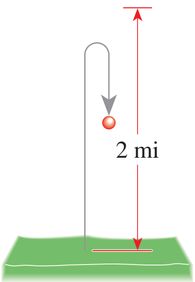
(b) Setting $h = 6400$ gives the equation



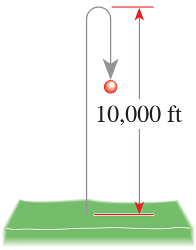
$$\begin{aligned} 6400 &= -16t^2 + 800t && \text{Set } h = 6400 \\ 16t^2 - 800t + 6400 &= 0 && \text{All terms to LHS} \\ t^2 - 50t + 400 &= 0 && \text{Divide by 16} \\ (t - 10)(t - 40) &= 0 && \text{Factor} \\ t = 10 \quad \text{or} \quad t &= 40 && \text{Solve} \end{aligned}$$

The bullet reaches 6400 ft after 10 s (on its ascent) and again after 40 s (on its descent to earth).

(c) Two miles is $2 \times 5280 = 10{,}560$ ft.



$$\begin{aligned} 10{,}560 &= -16t^2 + 800t && \text{Set } h = 10{,}560 \\ 16t^2 - 800t + 10{,}560 &= 0 && \text{All terms to LHS} \\ t^2 - 50t + 660 &= 0 && \text{Divide by 16} \end{aligned}$$



The discriminant of this equation is $D = (-50)^2 - 4(660) = -140$, which is negative. Thus the equation has no real solution. The bullet never reaches a height of 2 mi.

(d) Each height the bullet reaches is attained twice, once on its ascent and once on its descent. The only exception is the highest point of its path, which is reached only once. This means that for the highest value of h, the following equation has only one solution for t:

$$h = -16t^2 + 800t$$

$$16t^2 - 800t + h = 0 \qquad \text{All terms to LHS}$$

This in turn means that the discriminant D of the equation is 0, so

$$D = (-800)^2 - 4(16)h = 0$$

$$640{,}000 - 64h = 0$$

$$h = 10{,}000$$

The maximum height reached is 10,000 ft.

Now Try Exercise 129 ■

■ Other Types of Equations

So far we have learned how to solve linear and quadratic equations. Now we study other types of equations, including those that involve higher powers, fractional expressions, and radicals.

When we solve an equation that involves fractional expressions or radicals, we must be especially careful to check our final answers. The next two examples demonstrate why.

EXAMPLE 10 ■ An Equation Involving Fractional Expressions

Solve the equation $\dfrac{3}{x} - \dfrac{2}{x-3} = \dfrac{-12}{x^2 - 9}$.

SOLUTION We eliminate the denominators by multiplying each side by the lowest common denominator.

$$\left(\frac{3}{x} - \frac{2}{x-3}\right)x(x^2 - 9) = \frac{-12}{x^2 - 9}x(x^2 - 9) \qquad \text{Multiply by LCD, } x(x^2 - 9)$$

$$3(x^2 - 9) - 2x(x + 3) = -12x \qquad \text{Expand}$$

$$3x^2 - 27 - 2x^2 - 6x = -12x \qquad \text{Expand LHS}$$

$$x^2 - 6x - 27 = -12x \qquad \text{Add like terms on LHS}$$

$$x^2 + 6x - 27 = 0 \qquad \text{Add } 12x$$

$$(x - 3)(x + 9) = 0 \qquad \text{Factor}$$

$$x - 3 = 0 \quad \text{or} \quad x + 9 = 0 \qquad \text{Zero-Product Property}$$

$$x = 3 \qquad\qquad x = -9 \qquad \text{Solve}$$

CHECK YOUR ANSWERS

$x = 3$:

$$\text{LHS} = \frac{3}{3} - \frac{2}{3-3} \text{ undefined}$$

$$\text{RHS} = \frac{-12}{3^2 - 9} \text{ undefined} \quad ✗$$

$x = -9$:

$$\text{LHS} = \frac{3}{-9} - \frac{2}{-9-3} = -\frac{1}{6}$$

$$\text{RHS} = \frac{-12}{(-9)^2 - 9} = -\frac{1}{6}$$

$$\text{LHS} = \text{RHS} \quad ✓$$

We must check our answer because multiplying by an expression that contains the variable can introduce extraneous solutions. From *Check Your Answers* we see that the only solution is $x = -9$.

Now Try Exercise 89 ■

EXAMPLE 11 ■ An Equation Involving a Radical

Solve the equation $2x = 1 - \sqrt{2 - x}$.

CHECK YOUR ANSWERS

$x = -\frac{1}{4}$:

$$\text{LHS} = 2\left(-\tfrac{1}{4}\right) = -\tfrac{1}{2}$$
$$\text{RHS} = 1 - \sqrt{2 - \left(-\tfrac{1}{4}\right)} = 1 - \sqrt{\tfrac{9}{4}} = 1 - \tfrac{3}{2} = -\tfrac{1}{2}$$
$$\text{LHS} = \text{RHS} \quad ✓$$

$x = 1$:

$$\text{LHS} = 2(1) = 2$$
$$\text{RHS} = 1 - \sqrt{2 - 1} = 1 - 1 = 0$$
$$\text{LHS} \neq \text{RHS} \quad ✗$$

SOLUTION To eliminate the square root, we first isolate it on one side of the equal sign, then square.

$2x - 1 = -\sqrt{2 - x}$	Subtract 1
$(2x - 1)^2 = 2 - x$	Square each side
$4x^2 - 4x + 1 = 2 - x$	Expand LHS
$4x^2 - 3x - 1 = 0$	Add $-2 + x$
$(4x + 1)(x - 1) = 0$	Factor
$4x + 1 = 0$ or $x - 1 = 0$	Zero-Product Property
$x = -\frac{1}{4}$ $x = 1$	Solve

The values $x = -\frac{1}{4}$ and $x = 1$ are only potential solutions. We must check them to see whether they satisfy the original equation. From *Check Your Answers* we see that $x = -\frac{1}{4}$ is a solution but $x = 1$ is not. The only solution is $x = -\frac{1}{4}$.

Now Try Exercise 97 ■

When we solve an equation, we may end up with one or more **extraneous solutions**, that is, potential solutions that do not satisfy the original equation. In Example 10 the value $x = 3$ is an extraneous solution, and in Example 11 the value $x = 1$ is an extraneous solution. In the case of equations involving fractional expressions, potential solutions may be undefined in the original equation and hence become extraneous solutions. In the case of equations involving radicals, extraneous solutions may be introduced when we square each side of an equation because the operation of squaring can turn a false equation into a true one. For example, $-1 \neq 1$, but $(-1)^2 = 1^2$. Thus the squared equation may be true for more values of the variable than the original equation. That is why you must always check your answers to make sure that each satisfies the original equation.

An equation of the form $aW^2 + bW + c = 0$, where W is an algebraic expression, is an equation of **quadratic type**. We solve equations of quadratic type by substituting for the algebraic expression, as we see in the next two examples.

EXAMPLE 12 ■ A Fourth-Degree Equation of Quadratic Type

Find all solutions of the equation $x^4 - 8x^2 + 8 = 0$.

SOLUTION If we set $W = x^2$, then we get a quadratic equation in the new variable W.

$(x^2)^2 - 8x^2 + 8 = 0$	Write x^4 as $(x^2)^2$
$W^2 - 8W + 8 = 0$	Let $W = x^2$
$W = \dfrac{-(-8) \pm \sqrt{(-8)^2 - 4 \cdot 8}}{2} = 4 \pm 2\sqrt{2}$	Quadratic Formula
$x^2 = 4 \pm 2\sqrt{2}$	$W = x^2$
$x = \pm\sqrt{4 \pm 2\sqrt{2}}$	Take square roots

So there are four solutions:

$$\sqrt{4 + 2\sqrt{2}} \qquad \sqrt{4 - 2\sqrt{2}} \qquad -\sqrt{4 + 2\sqrt{2}} \qquad -\sqrt{4 - 2\sqrt{2}}$$

Using a calculator, we obtain the approximations $x \approx 2.61, 1.08, -2.61, -1.08$.

Now Try Exercise 103 ■

EXAMPLE 13 ■ An Equation Involving Fractional Powers

Find all solutions of the equation $x^{1/3} + x^{1/6} - 2 = 0$.

SOLUTION This equation is of quadratic type because if we let $W = x^{1/6}$, then $W^2 = (x^{1/6})^2 = x^{1/3}$.

$$
\begin{aligned}
x^{1/3} + x^{1/6} - 2 &= 0 && \\
W^2 + W - 2 &= 0 && \text{Let } W = x^{1/6} \\
(W - 1)(W + 2) &= 0 && \text{Factor} \\
W - 1 = 0 \quad \text{or} \quad W + 2 &= 0 && \text{Zero-Product Property} \\
W = 1 \qquad\qquad W &= -2 && \text{Solve} \\
x^{1/6} = 1 \qquad\qquad x^{1/6} &= -2 && W = x^{1/6} \\
x = 1^6 = 1 \qquad\qquad x &= (-2)^6 = 64 && \text{Take the 6th power}
\end{aligned}
$$

From *Check Your Answers* we see that $x = 1$ is a solution but $x = 64$ is not. The only solution is $x = 1$.

CHECK YOUR ANSWERS

$x = 1$:

$$\text{LHS} = 1^{1/3} + 1^{1/6} - 2 = 0$$

$$\text{RHS} = 0$$

$$\text{LHS} = \text{RHS} \quad ✓$$

$x = 64$:

$$\text{LHS} = 64^{1/3} + 64^{1/6} - 2 = 4 + 2 - 2 = 4$$

$$\text{RHS} = 0$$

$$\text{LHS} \neq \text{RHS} \quad ✗$$

Now Try Exercise 107

When solving an absolute value equation, we use the following property

$$|X| = C \quad \text{is equivalent to} \quad X = C \quad \text{or} \quad X = -C$$

where X is any algebraic expression. This property says that to solve an absolute value equation, we must solve two separate equations.

EXAMPLE 14 ■ An Absolute Value Equation

Solve the equation $|2x - 5| = 3$.

SOLUTION By the definition of absolute value, $|2x - 5| = 3$ is equivalent to

$$
\begin{aligned}
2x - 5 = 3 \qquad &\text{or} \qquad 2x - 5 = -3 \\
2x = 8 \qquad &\qquad\qquad 2x = 2 \\
x = 4 \qquad &\qquad\qquad x = 1
\end{aligned}
$$

The solutions are $x = 1$, $x = 4$.

Now Try Exercise 113

1.5 EXERCISES

CONCEPTS

1. *Yes or No*? If *No*, give a reason.

(a) When you add the same number to each side of an equation, do you always get an equivalent equation?

(b) When you multiply each side of an equation by the same nonzero number, do you always get an equivalent equation?

(c) When you square each side of an equation, do you always get an equivalent equation?

2. What is a logical first step in solving the equation?

(a) $(x + 5)^2 = 64$ **(b)** $(x + 5)^2 + 5 = 64$

(c) $x^2 + x = 2$

3. Explain how you would use each method to solve the equation $x^2 - 4x - 5 = 0$.

(a) By factoring: ________

(b) By completing the square: ________

(c) By using the Quadratic Formula: ________

4. **(a)** The solutions of the equation $x^2(x - 4) = 0$ are ________.

(b) To solve the equation $x^3 - 4x^2 = 0$, we ________ the left-hand side.

5. Solve the equation $\sqrt{2x} + x = 0$ by doing the following steps.

(a) Isolate the radical: ________________.

(b) Square both sides: ________________.

(c) The solutions of the resulting quadratic equation are ________.

(d) The solution(s) that satisfy the original equation are ________.

6. The equation $(x + 1)^2 - 5(x + 1) + 6 = 0$ is of ________ type. To solve the equation, we set $W =$ ________. The resulting quadratic equation is ________.

7. To eliminate the denominators in the equation $\dfrac{3}{x} + \dfrac{5}{x+2} = 2$, we multiply each side by the lowest common denominator ________________ to get the equivalent equation ________________.

8. To eliminate the square root in the equation $2x + 1 = \sqrt{x + 1}$, we ________________ each side to get the equation ________________.

SKILLS

9–12 ■ Solution? Determine whether the given value is a solution of the equation.

9. $4x + 7 = 9x - 3$

(a) $x = -2$ **(b)** $x = 2$

10. $1 - [2 - (3 - x)] = 4x - (6 + x)$

(a) $x = 2$ **(b)** $x = 4$

11. $\dfrac{1}{x} - \dfrac{1}{x - 4} = 1$

(a) $x = 2$ **(b)** $x = 4$

12. $\dfrac{x^{3/2}}{x - 6} = x - 8$

(a) $x = 4$ **(b)** $x = 8$

13–30 ■ Linear Equations The given equation is either linear or equivalent to a linear equation. Solve the equation.

13. $5x - 6 = 14$

14. $3x + 4 = 7$

15. $\frac{1}{2}x - 8 = 1$

16. $3 + \frac{1}{3}x = 5$

17. $-x + 3 = 4x$

18. $2x + 3 = 7 - 3x$

19. $\dfrac{x}{3} - 2 = \dfrac{5}{3}x + 7$

20. $\frac{2}{5}x - 1 = \frac{3}{10}x + 3$

21. $2(1 - x) = 3(1 + 2x) + 5$

22. $\dfrac{2}{3}y + \dfrac{1}{2}(y - 3) = \dfrac{y + 1}{4}$

23. $x - \frac{1}{3}x - \frac{1}{2}x - 5 = 0$

24. $2x - \dfrac{x}{2} + \dfrac{x + 1}{4} = 6x$

25. $\dfrac{1}{x} = \dfrac{4}{3x} + 1$

26. $\dfrac{2x - 1}{x + 2} = \dfrac{4}{5}$

27. $\dfrac{3}{x + 1} - \dfrac{1}{2} = \dfrac{1}{3x + 3}$

28. $\dfrac{4}{x - 1} + \dfrac{2}{x + 1} = \dfrac{35}{x^2 - 1}$

29. $(t - 4)^2 = (t + 4)^2 + 32$

30. $\sqrt{3}x + \sqrt{12} = \dfrac{x + 5}{\sqrt{3}}$

31–44 ■ Solving for a Variable Solve the equation for the indicated variable.

31. $PV = nRT$; for R

32. $F = G\dfrac{mM}{r^2}$; for m

33. $P = 2l + 2w$; for w

34. $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$; for R_1

35. $\dfrac{ax + b}{cx + d} = 2$; for x

36. $a - 2[b - 3(c - x)] = 6$; for x

37. $a^2x + (a - 1) = (a + 1)x$; for x

38. $\dfrac{a + 1}{b} = \dfrac{a - 1}{b} + \dfrac{b + 1}{a}$; for a

39. $V = \frac{1}{3}\pi r^2 h$; for r

40. $F = G\dfrac{mM}{r^2}$; for r

41. $a^2 + b^2 = c^2$; for b

42. $A = P\left(1 + \dfrac{i}{100}\right)^2$; for i

43. $h = \frac{1}{2}gt^2 + v_0 t$; for t

44. $S = \dfrac{n(n + 1)}{2}$; for n

45–56 ■ Solving by Factoring Find all real solutions of the equation by factoring.

45. $x^2 + x - 12 = 0$

46. $x^2 + 3x - 4 = 0$

47. $x^2 - 7x + 12 = 0$

48. $x^2 + 8x + 12 = 0$

49. $4x^2 - 4x - 15 = 0$

50. $2y^2 + 7y + 3 = 0$

51. $3x^2 + 5x = 2$

52. $6x(x - 1) = 21 - x$

53. $2x^2 = 8$

54. $3x^2 - 27 = 0$

55. $(2x - 5)^2 = 81$

56. $(5x + 1)^2 + 3 = 10$

57–64 ■ Completing the Square Find all real solutions of the equation by completing the square.

57. $x^2 + 2x - 5 = 0$

58. $x^2 - 4x + 2 = 0$

59. $x^2 - 6x - 11 = 0$

60. $x^2 + 3x - \frac{7}{4} = 0$

61. $2x^2 + 8x + 1 = 0$

62. $3x^2 - 6x - 1 = 0$

63. $4x^2 - x = 0$

64. $x^2 = \frac{3}{4}x - \frac{1}{8}$

65–80 ■ Quadratic Equations Find all real solutions of the quadratic equation.

65. $x^2 - 2x - 15 = 0$
66. $x^2 + 5x - 6 = 0$
67. $x^2 - 13x + 42 = 0$
68. $x^2 + 10x - 600 = 0$
69. $2x^2 + x - 3 = 0$
70. $3x^2 + 7x + 4 = 0$
71. $3x^2 + 6x - 5 = 0$
72. $x^2 - 6x + 1 = 0$
73. $9x^2 + 12x + 4 = 0$
74. $4x^2 - 4x + 1 = 0$
75. $4x^2 + 16x - 9 = 0$
76. $0 = x^2 - 4x + 1$
77. $7x^2 - 2x + 4 = 0$
78. $w^2 = 3(w - 1)$
79. $10y^2 - 16y + 5 = 0$
80. $25x^2 + 70x + 49 = 0$

81–86 ■ Discriminant Use the discriminant to determine the number of real solutions of the equation. Do not solve the equation.

81. $x^2 - 6x + 1 = 0$
82. $3x^2 = 6x - 9$
83. $x^2 + 2.20x + 1.21 = 0$
84. $x^2 + 2.21x + 1.21 = 0$
85. $4x^2 + 5x + \frac{13}{8} = 0$
86. $x^2 + rx - s = 0 \quad (s > 0)$

87–116 ■ Other Equations Find all real solutions of the equation.

87. $\dfrac{x^2}{x + 100} = 50$
88. $\dfrac{1}{x - 1} - \dfrac{2}{x^2} = 0$
89. $\dfrac{1}{x - 1} + \dfrac{1}{x + 2} = \dfrac{5}{4}$
90. $\dfrac{x + 5}{x - 2} = \dfrac{5}{x + 2} + \dfrac{28}{x^2 - 4}$
91. $\dfrac{10}{x} - \dfrac{12}{x - 3} + 4 = 0$
92. $\dfrac{x}{2x + 7} - \dfrac{x + 1}{x + 3} = 1$
93. $5 = \sqrt{4x - 3}$
94. $\sqrt{8x - 1} = 3$
95. $\sqrt{2x - 1} = \sqrt{3x - 5}$
96. $\sqrt{3 + x} = \sqrt{x^2 + 1}$
97. $\sqrt{2x + 1} + 1 = x$
98. $\sqrt{5 - x} + 1 = x - 2$
99. $2x + \sqrt{x + 1} = 8$
100. $x - \sqrt{9 - 3x} = 0$
101. $\sqrt{3x + 1} = 2 + \sqrt{x + 1}$
102. $\sqrt{1 + x} + \sqrt{1 - x} = 2$
103. $x^4 - 13x^2 + 40 = 0$
104. $x^4 - 5x^2 + 4 = 0$
105. $2x^4 + 4x^2 + 1 = 0$
106. $x^6 - 2x^3 - 3 = 0$
107. $x^{4/3} - 5x^{2/3} + 6 = 0$
108. $\sqrt{x} - 3\sqrt[4]{x} - 4 = 0$
109. $4(x + 1)^{1/2} - 5(x + 1)^{3/2} + (x + 1)^{5/2} = 0$
110. $x^{1/2} + 3x^{-1/2} = 10x^{-3/2}$
111. $x^{1/2} - 3x^{1/3} = 3x^{1/6} - 9$
112. $x - 5\sqrt{x} + 6 = 0$
113. $|3x + 5| = 1$
114. $|2x| = 3$
115. $|x - 4| = 0.01$
116. $|x - 6| = -1$

SKILLS Plus

117–122 ■ More on Solving Equations Find all real solutions of the equation.

117. $\dfrac{1}{x^3} + \dfrac{4}{x^2} + \dfrac{4}{x} = 0$
118. $4x^{-4} - 16x^{-2} + 4 = 0$
119. $\sqrt{\sqrt{x + 5} + x} = 5$
120. $\sqrt[3]{4x^2 - 4x} = x$
121. $x^2\sqrt{x + 3} = (x + 3)^{3/2}$
122. $\sqrt{11 - x^2} - \dfrac{2}{\sqrt{11 - x^2}} = 1$

123–126 ■ More on Solving Equations Solve the equation for the variable x. The constants a and b represent positive real numbers.

123. $x^4 - 5ax^2 + 4a^2 = 0$
124. $a^3x^3 + b^3 = 0$
125. $\sqrt{x + a} + \sqrt{x - a} = \sqrt{2}\sqrt{x + 6}$
126. $\sqrt{x} - a\sqrt[3]{x} + b\sqrt[6]{x} - ab = 0$

APPLICATIONS

127–128 ■ Falling-Body Problems Suppose an object is dropped from a height h_0 above the ground. Then its height after t seconds is given by $h = -16t^2 + h_0$, where h is measured in feet. Use this information to solve the problem.

127. If a ball is dropped from 288 ft above the ground, how long does it take to reach ground level?

128. A ball is dropped from the top of a building 96 ft tall.
   (a) How long will it take to fall half the distance to ground level?
   (b) How long will it take to fall to ground level?

129–130 ■ Falling-Body Problems Use the formula $h = -16t^2 + v_0t$ discussed in Example 9.

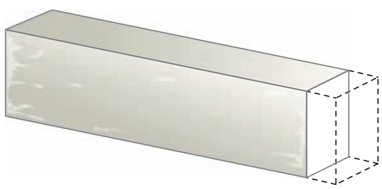
129. A ball is thrown straight upward at an initial speed of $v_0 = 40$ ft/s.
   (a) When does the ball reach a height of 24 ft?
   (b) When does it reach a height of 48 ft?
   (c) What is the greatest height reached by the ball?
   (d) When does the ball reach the highest point of its path?
   (e) When does the ball hit the ground?

130. How fast would a ball have to be thrown upward to reach a maximum height of 100 ft? [*Hint:* Use the discriminant of the equation $16t^2 - v_0t + h = 0$.]

131. **Shrinkage in Concrete Beams** As concrete dries, it shrinks—the higher the water content, the greater the shrinkage. If a concrete beam has a water content of w kg/m^3, then it will shrink by a factor

$$S = \frac{0.032w - 2.5}{10{,}000}$$

where S is the fraction of the original beam length that disappears due to shrinkage.
   (a) A beam 12.025 m long is cast in concrete that contains 250 kg/m^3 water. What is the shrinkage factor S? How long will the beam be when it has dried?
   (b) A beam is 10.014 m long when wet. We want it to shrink to 10.009 m, so the shrinkage factor should be $S = 0.00050$. What water content will provide this amount of shrinkage?



132. The Lens Equation If F is the focal length of a convex lens and an object is placed at a distance x from the lens, then its image will be at a distance y from the lens, where F, x, and y are related by the *lens equation*

$$\frac{1}{F} = \frac{1}{x} + \frac{1}{y}$$

Suppose that a lens has a focal length of 4.8 cm and that the image of an object is 4 cm closer to the lens than the object itself. How far from the lens is the object?

133. Fish Population The fish population in a certain lake rises and falls according to the formula

$$F = 1000(30 + 17t - t^2)$$

Here F is the number of fish at time t, where t is measured in years since January 1, 2002, when the fish population was first estimated.

(a) On what date will the fish population again be the same as it was on January 1, 2002?

(b) By what date will all the fish in the lake have died?

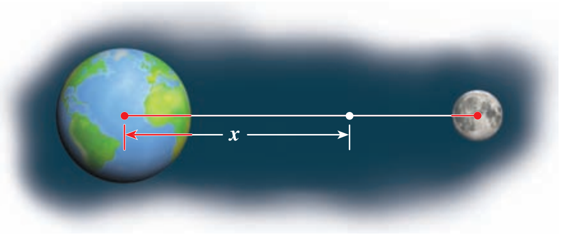
134. Fish Population A large pond is stocked with fish. The fish population P is modeled by the formula $P = 3t + 10\sqrt{t} + 140$, where t is the number of days since the fish were first introduced into the pond. How many days will it take for the fish population to reach 500?

135. Profit A small-appliance manufacturer finds that the profit P (in dollars) generated by producing x microwave ovens per week is given by the formula $P = \frac{1}{10}x(300 - x)$, provided that $0 \le x \le 200$. How many ovens must be manufactured in a given week to generate a profit of $1250?

136. Gravity If an imaginary line segment is drawn between the centers of the earth and the moon, then the net gravitational force F acting on an object situated on this line segment is

$$F = \frac{-K}{x^2} + \frac{0.012K}{(239 - x)^2}$$

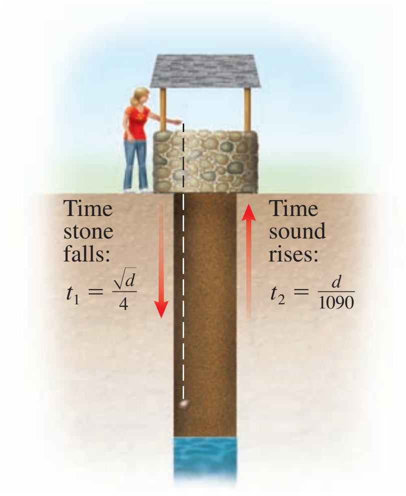
where $K > 0$ is a constant and x is the distance of the object from the center of the earth, measured in thousands of miles. How far from the center of the earth is the "dead spot" where no net gravitational force acts upon the object? (Express your answer to the nearest thousand miles.)



137. Depth of a Well One method for determining the depth of a well is to drop a stone into it and then measure the time it takes until the splash is heard. If d is the depth of the well (in feet) and t_1 the time (in seconds) it takes for the stone to fall, then $d = 16t_1^2$, so $t_1 = \sqrt{d}/4$. Now if t_2 is the time it takes for the sound to travel back up, then $d = 1090t_2$ because the speed of sound is 1090 ft/s. So $t_2 = d/1090$. Thus the total time elapsed between dropping the stone and hearing the splash is

$$t_1 + t_2 = \frac{\sqrt{d}}{4} + \frac{d}{1090}$$

How deep is the well if this total time is 3 s?



DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

138. DISCUSS: A Family of Equations The equation

$$3x + k - 5 = kx - k + 1$$

is really a **family of equations**, because for each value of k, we get a different equation with the unknown x. The letter k is called a **parameter** for this family. What value should we pick for k to make the given value of x a solution of the resulting equation?

(a) $x = 0$ **(b)** $x = 1$ **(c)** $x = 2$

139. DISCUSS: Proof That 0 = 1? The following steps appear to give equivalent equations, which seem to prove that $1 = 0$. Find the error.

$x = 1$	Given
$x^2 = x$	Multiply by x
$x^2 - x = 0$	Subtract x
$x(x - 1) = 0$	Factor
$\dfrac{x(x-1)}{x-1} = \dfrac{0}{x-1}$	Divide by $x - 1$
$x = 0$	Simplify
$1 = 0$	Given $x = 1$

140. DISCOVER ■ PROVE: Relationship Between Solutions and Coefficients The Quadratic Formula gives us the solutions