1.4 RATIONAL EXPRESSIONS

■ The Domain of an Algebraic Expression ■ Simplifying Rational Expressions ■ Multiplying and Dividing Rational Expressions ■ Adding and Subtracting Rational Expressions ■ Compound Fractions ■ Rationalizing the Denominator or the Numerator ■ Avoiding Common Errors

A quotient of two algebraic expressions is called a **fractional expression**. Here are some examples:

$$\frac{2x}{x-1} \qquad \frac{y-2}{y^2+4} \qquad \frac{x^3-x}{x^2-5x+6} \qquad \frac{x}{\sqrt{x^2+1}}$$

A **rational expression** is a fractional expression in which both the numerator and the denominator are polynomials. For example, the first three expressions in the above list are rational expressions, but the fourth is not, since its denominator contains a radical. In this section we learn how to perform algebraic operations on rational expressions.

■ The Domain of an Algebraic Expression

Expression	Domain
$\frac{1}{x}$	$\{x \mid x \neq 0\}$
$\sqrt{x}$	$\{x \mid x \geq 0\}$
$\frac{1}{\sqrt{x}}$	$\{x \mid x > 0\}$

In general, an algebraic expression may not be defined for all values of the variable. The **domain** of an algebraic expression is the set of real numbers that the variable is permitted to have. The table in the margin gives some basic expressions and their domains.

EXAMPLE 1 ■ Finding the Domain of an Expression

Find the domains of the following expressions.

(a) $2x^2 + 3x - 1$ **(b)** $\frac{x}{x^2 - 5x + 6}$ **(c)** $\frac{\sqrt{x}}{x-5}$

SOLUTION

(a) This polynomial is defined for every x. Thus the domain is the set $\mathbb{R}$ of real numbers.

(b) We first factor the denominator.

$$\frac{x}{x^2-5x+6} = \frac{x}{(x-2)(x-3)}$$

Denominator would be 0 if $x = 2$ or $x = 3$

Since the denominator is zero when $x = 2$ or 3, the expression is not defined for these numbers. The domain is $\{x \mid x \neq 2 \text{ and } x \neq 3\}$.

(c) For the numerator to be defined, we must have $x \geq 0$. Also, we cannot divide by zero, so $x \neq 5$.

Must have $x \geq 0$ to take square root

$$\frac{\sqrt{x}}{x-5}$$

Denominator would be 0 if $x = 5$

Thus the domain is $\{x \mid x \geq 0 \text{ and } x \neq 5\}$.

Now Try Exercise 13 ■

Simplifying Rational Expressions

To **simplify rational expressions**, we factor both numerator and denominator and use the following property of fractions:

$$\frac{AC}{BC} = \frac{A}{B}$$

This allows us to **cancel** common factors from the numerator and denominator.

EXAMPLE 2 ■ Simplifying Rational Expressions by Cancellation

Simplify: $\dfrac{x^2 - 1}{x^2 + x - 2}$

SOLUTION

We can't cancel the x^2's in $\dfrac{x^2 - 1}{x^2 + x - 2}$ because x^2 is not a factor.

$$\begin{aligned}\frac{x^2 - 1}{x^2 + x - 2} &= \frac{(x - 1)(x + 1)}{(x - 1)(x + 2)} && \text{Factor} \\ &= \frac{x + 1}{x + 2} && \text{Cancel common factors}\end{aligned}$$

Now Try Exercise 19

Multiplying and Dividing Rational Expressions

To **multiply rational expressions**, we use the following property of fractions:

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}$$

This says that to multiply two fractions, we multiply their numerators and multiply their denominators.

EXAMPLE 3 ■ Multiplying Rational Expressions

Perform the indicated multiplication and simplify: $\dfrac{x^2 + 2x - 3}{x^2 + 8x + 16} \cdot \dfrac{3x + 12}{x - 1}$

SOLUTION We first factor.

$$\begin{aligned}\frac{x^2 + 2x - 3}{x^2 + 8x + 16} \cdot \frac{3x + 12}{x - 1} &= \frac{(x - 1)(x + 3)}{(x + 4)^2} \cdot \frac{3(x + 4)}{x - 1} && \text{Factor} \\ &= \frac{3(x - 1)(x + 3)(x + 4)}{(x - 1)(x + 4)^2} && \text{Property of fractions} \\ &= \frac{3(x + 3)}{x + 4} && \text{Cancel common factors}\end{aligned}$$

Now Try Exercise 27

To **divide rational expressions**, we use the following property of fractions:

$$\boxed{\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C}}$$

This says that to divide a fraction by another fraction, we invert the divisor and multiply.

EXAMPLE 4 ■ Dividing Rational Expressions

Perform the indicated division and simplify: $\dfrac{x-4}{x^2-4} \div \dfrac{x^2-3x-4}{x^2+5x+6}$

SOLUTION

$$\frac{x-4}{x^2-4} \div \frac{x^2-3x-4}{x^2+5x+6} = \frac{x-4}{x^2-4} \cdot \frac{x^2+5x+6}{x^2-3x-4} \qquad \text{Invert and multiply}$$

$$= \frac{(x-4)(x+2)(x+3)}{(x-2)(x+2)(x-4)(x+1)} \qquad \text{Factor}$$

$$= \frac{x+3}{(x-2)(x+1)} \qquad \text{Cancel common factors}$$

Now Try Exercise 33 ■

■ Adding and Subtracting Rational Expressions

Avoid making the following error:

$$\frac{A}{B+C} \cancel{=} \frac{A}{B} + \frac{A}{C}$$

For instance, if we let $A = 2$, $B = 1$, and $C = 1$, then we see the error:

$$\frac{2}{1+1} \stackrel{?}{=} \frac{2}{1} + \frac{2}{1}$$

$$\frac{2}{2} \stackrel{?}{=} 2 + 2$$

$$1 \stackrel{?}{=} 4 \qquad \text{Wrong!}$$

To **add or subtract rational expressions**, we first find a common denominator and then use the following property of fractions:

$$\boxed{\frac{A}{C} + \frac{B}{C} = \frac{A+B}{C}}$$

Although any common denominator will work, it is best to use the **least common denominator** (LCD) as explained in Section 1.1. The LCD is found by factoring each denominator and taking the product of the distinct factors, using the highest power that appears in any of the factors.

EXAMPLE 5 ■ Adding and Subtracting Rational Expressions

Perform the indicated operations and simplify.

(a) $\dfrac{3}{x-1} + \dfrac{x}{x+2}$ **(b)** $\dfrac{1}{x^2-1} - \dfrac{2}{(x+1)^2}$

SOLUTION

(a) Here the LCD is simply the product $(x-1)(x+2)$.

$$\frac{3}{x-1} + \frac{x}{x+2} = \frac{3(x+2)}{(x-1)(x+2)} + \frac{x(x-1)}{(x-1)(x+2)} \qquad \text{Write fractions using LCD}$$

$$= \frac{3x+6+x^2-x}{(x-1)(x+2)} \qquad \text{Add fractions}$$

$$= \frac{x^2+2x+6}{(x-1)(x+2)} \qquad \text{Combine terms in numerator}$$

(b) The LCD of $x^2 - 1 = (x - 1)(x + 1)$ and $(x + 1)^2$ is $(x - 1)(x + 1)^2$.

$$
\begin{aligned}
\frac{1}{x^2 - 1} - \frac{2}{(x + 1)^2} &= \frac{1}{(x - 1)(x + 1)} - \frac{2}{(x + 1)^2} && \text{Factor} \\
&= \frac{(x + 1) - 2(x - 1)}{(x - 1)(x + 1)^2} && \text{Combine fractions using LCD} \\
&= \frac{x + 1 - 2x + 2}{(x - 1)(x + 1)^2} && \text{Distributive Property} \\
&= \frac{3 - x}{(x - 1)(x + 1)^2} && \text{Combine terms in numerator}
\end{aligned}
$$

Now Try Exercises 43 and 45 ■

■ Compound Fractions

A **compound fraction** is a fraction in which the numerator, the denominator, or both, are themselves fractional expressions.

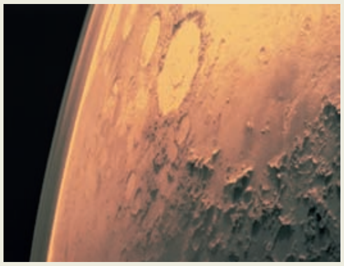
EXAMPLE 6 ■ Simplifying a Compound Fraction

Simplify: $\dfrac{\dfrac{x}{y} + 1}{1 - \dfrac{y}{x}}$

SOLUTION 1 We combine the terms in the numerator into a single fraction. We do the same in the denominator. Then we invert and multiply.

$$
\begin{aligned}
\frac{\dfrac{x}{y} + 1}{1 - \dfrac{y}{x}} &= \frac{\dfrac{x + y}{y}}{\dfrac{x - y}{x}} = \frac{x + y}{y} \cdot \frac{x}{x - y} \\
&= \frac{x(x + y)}{y(x - y)}
\end{aligned}
$$

Mathematics in the Modern World



Courtesy of NASA

Error-Correcting Codes

The pictures sent back by the *Pathfinder* spacecraft from the surface of Mars on July 4, 1997, were astoundingly clear. But few viewing these pictures were aware of the complex mathematics used to accomplish that feat. The distance to Mars is enormous, and the background noise (or static) is many times stronger than the original signal emitted by the spacecraft. So when scientists receive the signal, it is full of errors. To get a clear picture, the errors must be found and corrected. This same problem of errors is routinely encountered in transmitting bank records when you use an ATM machine or voice when you are talking on the telephone.

To understand how errors are found and corrected, we must first understand that to transmit pictures, sound, or text, we transform them into bits (the digits 0 or 1; see page 28). To help the receiver recognize errors, the message is "coded" by inserting additional bits. For example, suppose you want to transmit the message "10100." A very simple-minded code is as follows: Send each digit a million times. The person receiving the message reads it in blocks of a million digits. If the first block is mostly 1's, he concludes that you are probably trying to transmit a 1, and so on. To say that this code is not efficient is a bit of an understatement; it requires sending a million times more data than the original message. Another method inserts "check digits." For example, for each block of eight digits insert a ninth digit; the inserted digit is 0 if there is an even number of 1's in the block and 1 if there is an odd number. So if a single digit is wrong (a 0 changed to a 1 or vice versa), the check digits allow us to recognize that an error has occurred. This method does not tell us where the error is, so we can't correct it. Modern error-correcting codes use interesting mathematical algorithms that require inserting relatively few digits but that allow the receiver to not only recognize, but also correct, errors. The first error-correcting code was developed in the 1940s by Richard Hamming at MIT. It is interesting to note that the English language has a built-in error correcting mechanism; to test it, try reading this error-laden sentence: Gve mo libty ox giv ne deth.

SOLUTION 2 We find the LCD of all the fractions in the expression, then multiply numerator and denominator by it. In this example the LCD of all the fractions is xy. Thus

$$\frac{\frac{x}{y}+1}{1-\frac{y}{x}} = \frac{\frac{x}{y}+1}{1-\frac{y}{x}} \cdot \frac{xy}{xy} \quad \text{Multiply numerator and denominator by } xy$$

$$= \frac{x^2+xy}{xy-y^2} \quad \text{Simplify}$$

$$= \frac{x(x+y)}{y(x-y)} \quad \text{Factor}$$

Now Try Exercises 59 and 65 ■

The next two examples show situations in calculus that require the ability to work with fractional expressions.

EXAMPLE 7 ■ Simplifying a Compound Fraction

Simplify: $\dfrac{\frac{1}{a+h}-\frac{1}{a}}{h}$

SOLUTION We begin by combining the fractions in the numerator using a common denominator.

We can also simplify by multiplying the numerator and the denominator by $a(a+h)$.

$$\frac{\frac{1}{a+h}-\frac{1}{a}}{h} = \frac{\frac{a-(a+h)}{a(a+h)}}{h} \quad \text{Combine fractions in the numerator}$$

$$= \frac{a-(a+h)}{a(a+h)} \cdot \frac{1}{h} \quad \text{Property 2 of fractions (invert divisor and multiply)}$$

$$= \frac{a-a-h}{a(a+h)} \cdot \frac{1}{h} \quad \text{Distributive Property}$$

$$= \frac{-h}{a(a+h)} \cdot \frac{1}{h} \quad \text{Simplify}$$

$$= \frac{-1}{a(a+h)} \quad \text{Property 5 of fractions (cancel common factors)}$$

Now Try Exercise 73 ■

EXAMPLE 8 ■ Simplifying a Compound Fraction

Simplify: $\dfrac{(1+x^2)^{1/2}-x^2(1+x^2)^{-1/2}}{1+x^2}$

SOLUTION 1 Factor $(1+x^2)^{-1/2}$ from the numerator.

Factor out the power of $1+x^2$ with the *smallest* exponent, in this case $(1+x^2)^{-1/2}$.

$$\frac{(1+x^2)^{1/2}-x^2(1+x^2)^{-1/2}}{1+x^2} = \frac{(1+x^2)^{-1/2}[(1+x^2)-x^2]}{1+x^2}$$

$$= \frac{(1+x^2)^{-1/2}}{1+x^2} = \frac{1}{(1+x^2)^{3/2}}$$

SOLUTION 2 Since $(1 + x^2)^{-1/2} = 1/(1 + x^2)^{1/2}$ is a fraction, we can clear all fractions by multiplying numerator and denominator by $(1 + x^2)^{1/2}$.

$$\frac{(1 + x^2)^{1/2} - x^2(1 + x^2)^{-1/2}}{1 + x^2} = \frac{(1 + x^2)^{1/2} - x^2(1 + x^2)^{-1/2}}{1 + x^2} \cdot \frac{(1 + x^2)^{1/2}}{(1 + x^2)^{1/2}}$$

$$= \frac{(1 + x^2) - x^2}{(1 + x^2)^{3/2}} = \frac{1}{(1 + x^2)^{3/2}}$$

Now Try Exercise 81 ■

■ Rationalizing the Denominator or the Numerator

If a fraction has a denominator of the form $A + B\sqrt{C}$, we can rationalize the denominator by multiplying numerator and denominator by the **conjugate radical** $A - B\sqrt{C}$. This works because, by Special Product Formula 1 in Section 1.3, the product of the denominator and its conjugate radical does not contain a radical:

$$(A + B\sqrt{C})(A - B\sqrt{C}) = A^2 - B^2C$$

EXAMPLE 9 ■ Rationalizing the Denominator

Rationalize the denominator: $\dfrac{1}{1 + \sqrt{2}}$

SOLUTION We multiply both the numerator and the denominator by the conjugate radical of $1 + \sqrt{2}$, which is $1 - \sqrt{2}$.

$$\frac{1}{1 + \sqrt{2}} = \frac{1}{1 + \sqrt{2}} \cdot \frac{1 - \sqrt{2}}{1 - \sqrt{2}}$$ Multiply numerator and denominator by the conjugate radical

$$= \frac{1 - \sqrt{2}}{1^2 - (\sqrt{2})^2}$$ Special Product Formula 1

Special Product Formula 1
$(A + B)(A - B) = A^2 - B^2$

$$= \frac{1 - \sqrt{2}}{1 - 2} = \frac{1 - \sqrt{2}}{-1} = \sqrt{2} - 1$$

Now Try Exercise 85 ■

EXAMPLE 10 ■ Rationalizing the Numerator

Rationalize the numerator: $\dfrac{\sqrt{4 + h} - 2}{h}$

SOLUTION We multiply numerator and denominator by the conjugate radical $\sqrt{4 + h} + 2$.

$$\frac{\sqrt{4 + h} - 2}{h} = \frac{\sqrt{4 + h} - 2}{h} \cdot \frac{\sqrt{4 + h} + 2}{\sqrt{4 + h} + 2}$$ Multiply numerator and denominator by the conjugate radical

$$= \frac{(\sqrt{4 + h})^2 - 2^2}{h(\sqrt{4 + h} + 2)}$$ Special Product Formula 1

Special Product Formula 1
$(A + B)(A - B) = A^2 - B^2$

$$= \frac{4 + h - 4}{h(\sqrt{4 + h} + 2)}$$

$$= \frac{h}{h(\sqrt{4 + h} + 2)} = \frac{1}{\sqrt{4 + h} + 2}$$ Property 5 of fractions (cancel common factors)

Now Try Exercise 91 ■

Avoiding Common Errors

Don't make the mistake of applying properties of multiplication to the operation of addition. Many of the common errors in algebra involve doing just that. The following table states several properties of multiplication and illustrates the error in applying them to addition.

Correct multiplication property	Common error with addition
$(a \cdot b)^2 = a^2 \cdot b^2$	$(a + b)^2 \cancel{=} a^2 + b^2$
$\sqrt{a \cdot b} = \sqrt{a}\sqrt{b} \quad (a, b \geq 0)$	$\sqrt{a + b} \cancel{=} \sqrt{a} + \sqrt{b}$
$\sqrt{a^2 \cdot b^2} = a \cdot b \quad (a, b \geq 0)$	$\sqrt{a^2 + b^2} \cancel{=} a + b$
$\dfrac{1}{a} \cdot \dfrac{1}{b} = \dfrac{1}{a \cdot b}$	$\dfrac{1}{a} + \dfrac{1}{b} \cancel{=} \dfrac{1}{a + b}$
$\dfrac{ab}{a} = b$	$\dfrac{a + b}{a} \cancel{=} b$
$a^{-1} \cdot b^{-1} = (a \cdot b)^{-1}$	$a^{-1} + b^{-1} \cancel{=} (a + b)^{-1}$

To verify that the equations in the right-hand column are wrong, simply substitute numbers for a and b and calculate each side. For example, if we take $a = 2$ and $b = 2$ in the fourth error, we get different values for the left- and right-hand sides:

$$\underbrace{\frac{1}{a} + \frac{1}{b} = \frac{1}{2} + \frac{1}{2} = 1}_{\text{Left-hand side}} \qquad \underbrace{\frac{1}{a + b} = \frac{1}{2 + 2} = \frac{1}{4}}_{\text{Right-hand side}}$$

Since $1 \neq \frac{1}{4}$, the stated equation is wrong. You should similarly convince yourself of the error in each of the other equations. (See Exercises 101 and 102.)

1.4 EXERCISES

CONCEPTS

1. Which of the following are rational expressions?

(a) $\dfrac{3x}{x^2 - 1}$ **(b)** $\dfrac{\sqrt{x + 1}}{2x + 3}$ **(c)** $\dfrac{x(x^2 - 1)}{x + 3}$

2. To simplify a rational expression, we cancel *factors* that are common to the ________ and ________. So the expression

$$\frac{(x + 1)(x + 2)}{(x + 3)(x + 2)}$$

simplifies to ________.

3. To multiply two rational expressions, we multiply their ________ together and multiply their ________ together. So $\dfrac{2}{x + 1} \cdot \dfrac{x}{x + 3}$ is the same as ________.

4. Consider the expression $\dfrac{1}{x} - \dfrac{2}{x + 1} - \dfrac{x}{(x + 1)^2}$.

(a) How many terms does this expression have?

(b) Find the least common denominator of all the terms.

(c) Perform the addition and simplify.

5–6 ■ *Yes or No*? If *No*, give a reason. (Disregard any value that makes a denominator zero.)

5. (a) Is the expression $\dfrac{x(x + 1)}{(x + 1)^2}$ equal to $\dfrac{x}{x + 1}$?

(b) Is the expression $\sqrt{x^2 + 25}$ equal to $x + 5$?

6. (a) Is the expression $\dfrac{3 + a}{3}$ equal to $1 + \dfrac{a}{3}$?

(b) Is the expression $\dfrac{2}{4 + x}$ equal to $\dfrac{1}{2} + \dfrac{2}{x}$?

SKILLS

7–14 ■ Domain Find the domain of the expression.

7. $4x^2 - 10x + 3$

8. $-x^4 + x^3 + 9x$

9. $\dfrac{x^2 - 1}{x - 3}$

10. $\dfrac{2t^2 - 5}{3t + 6}$

11. $\sqrt{x + 3}$

12. $\dfrac{1}{\sqrt{x - 1}}$

13. $\dfrac{x^2 + 1}{x^2 - x - 2}$

14. $\dfrac{\sqrt{2x}}{x + 1}$

15–24 ■ Simplify Simplify the rational expression.

15. $\dfrac{5(x-3)(2x+1)}{10(x-3)^2}$ **16.** $\dfrac{4(x^2-1)}{12(x+2)(x-1)}$

17. $\dfrac{x-2}{x^2-4}$ **18.** $\dfrac{x^2-x-2}{x^2-1}$

19. $\dfrac{x^2+5x+6}{x^2+8x+15}$ **20.** $\dfrac{x^2-x-12}{x^2+5x+6}$

21. $\dfrac{y^2+y}{y^2-1}$ **22.** $\dfrac{y^2-3y-18}{2y^2+7y+3}$

23. $\dfrac{2x^3-x^2-6x}{2x^2-7x+6}$ **24.** $\dfrac{1-x^2}{x^3-1}$

25–38 ■ Multiply or Divide Perform the multiplication or division and simplify.

25. $\dfrac{4x}{x^2-4}\cdot\dfrac{x+2}{16x}$ **26.** $\dfrac{x^2-25}{x^2-16}\cdot\dfrac{x+4}{x+5}$

27. $\dfrac{x^2+2x-15}{x^2-25}\cdot\dfrac{x-5}{x+2}$ **28.** $\dfrac{x^2+2x-3}{x^2-2x-3}\cdot\dfrac{3-x}{3+x}$

29. $\dfrac{t-3}{t^2+9}\cdot\dfrac{t+3}{t^2-9}$ **30.** $\dfrac{x^2-x-6}{x^2+2x}\cdot\dfrac{x^3+x^2}{x^2-2x-3}$

31. $\dfrac{x^2+7x+12}{x^2+3x+2}\cdot\dfrac{x^2+5x+6}{x^2+6x+9}$

32. $\dfrac{x^2+2xy+y^2}{x^2-y^2}\cdot\dfrac{2x^2-xy-y^2}{x^2-xy-2y^2}$

33. $\dfrac{x+3}{4x^2-9}\div\dfrac{x^2+7x+12}{2x^2+7x-15}$

34. $\dfrac{2x+1}{2x^2+x-15}\div\dfrac{6x^2-x-2}{x+3}$

35. $\dfrac{\dfrac{x^3}{x+1}}{\dfrac{x}{x^2+2x+1}}$ **36.** $\dfrac{\dfrac{2x^2-3x-2}{x^2-1}}{\dfrac{2x^2+5x+2}{x^2+x-2}}$

37. $\dfrac{x/y}{z}$ **38.** $\dfrac{x}{y/z}$

39–58 ■ Add or Subtract Perform the addition or subtraction and simplify.

39. $1+\dfrac{1}{x+3}$ **40.** $\dfrac{3x-2}{x+1}-2$

41. $\dfrac{1}{x+5}+\dfrac{2}{x-3}$ **42.** $\dfrac{1}{x+1}+\dfrac{1}{x-1}$

43. $\dfrac{3}{x+1}-\dfrac{1}{x+2}$ **44.** $\dfrac{x}{x-4}-\dfrac{3}{x+6}$

45. $\dfrac{5}{2x-3}-\dfrac{3}{(2x-3)^2}$ **46.** $\dfrac{x}{(x+1)^2}+\dfrac{2}{x+1}$

47. $u+1+\dfrac{u}{u+1}$ **48.** $\dfrac{2}{a^2}-\dfrac{3}{ab}+\dfrac{4}{b^2}$

49. $\dfrac{1}{x^2}+\dfrac{1}{x^2+x}$ **50.** $\dfrac{1}{x}+\dfrac{1}{x^2}+\dfrac{1}{x^3}$

51. $\dfrac{2}{x+3}-\dfrac{1}{x^2+7x+12}$ **52.** $\dfrac{x}{x^2-4}+\dfrac{1}{x-2}$

53. $\dfrac{1}{x+3}+\dfrac{1}{x^2-9}$

54. $\dfrac{x}{x^2+x-2}-\dfrac{2}{x^2-5x+4}$

55. $\dfrac{2}{x}+\dfrac{3}{x-1}-\dfrac{4}{x^2-x}$

56. $\dfrac{x}{x^2-x-6}-\dfrac{1}{x+2}-\dfrac{2}{x-3}$

57. $\dfrac{1}{x^2+3x+2}-\dfrac{1}{x^2-2x-3}$

58. $\dfrac{1}{x+1}-\dfrac{2}{(x+1)^2}+\dfrac{3}{x^2-1}$

59–72 ■ Compound Fractions Simplify the compound fractional expression.

59. $\dfrac{1+\dfrac{1}{x}}{\dfrac{1}{x}-2}$ **60.** $\dfrac{1-\dfrac{2}{y}}{\dfrac{3}{y}-1}$

61. $\dfrac{1+\dfrac{1}{x+2}}{1-\dfrac{1}{x+2}}$ **62.** $\dfrac{1+\dfrac{1}{c-1}}{1-\dfrac{1}{c-1}}$

63. $\dfrac{\dfrac{1}{x-1}+\dfrac{1}{x+3}}{x+1}$ **64.** $\dfrac{\dfrac{x-3}{x-4}-\dfrac{x+2}{x+1}}{x+3}$

65. $\dfrac{x-\dfrac{x}{y}}{y-\dfrac{y}{x}}$ **66.** $\dfrac{x+\dfrac{y}{x}}{y+\dfrac{x}{y}}$

67. $\dfrac{\dfrac{x}{y}-\dfrac{y}{x}}{\dfrac{1}{x^2}-\dfrac{1}{y^2}}$ **68.** $x-\dfrac{y}{\dfrac{x}{y}+\dfrac{y}{x}}$

69. $\dfrac{x^{-2}-y^{-2}}{x^{-1}+y^{-1}}$ **70.** $\dfrac{x^{-1}+y^{-1}}{(x+y)^{-1}}$

71. $1-\dfrac{1}{1-\dfrac{1}{x}}$ **72.** $1+\dfrac{1}{1+\dfrac{1}{1+x}}$

73–78 ■ Expressions Found in Calculus Simplify the fractional expression. (Expressions like these arise in calculus.)

73. $\dfrac{\dfrac{1}{1+x+h}-\dfrac{1}{1+x}}{h}$ **74.** $\dfrac{\dfrac{1}{\sqrt{x+h}}-\dfrac{1}{\sqrt{x}}}{h}$

75. $\dfrac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h}$

76. $\dfrac{(x+h)^3 - 7(x+h) - (x^3 - 7x)}{h}$

77. $\sqrt{1 + \left(\dfrac{x}{\sqrt{1-x^2}}\right)^2}$ **78.** $\sqrt{1 + \left(x^3 - \dfrac{1}{4x^3}\right)^2}$

79–84 ■ Expressions Found in Calculus Simplify the expression. (This type of expression arises in calculus when using the "quotient rule.")

79. $\dfrac{3(x+2)^2(x-3)^2 - (x+2)^3(2)(x-3)}{(x-3)^4}$

80. $\dfrac{2x(x+6)^4 - x^2(4)(x+6)^3}{(x+6)^8}$

81. $\dfrac{2(1+x)^{1/2} - x(1+x)^{-1/2}}{x+1}$

82. $\dfrac{(1-x^2)^{1/2} + x^2(1-x^2)^{-1/2}}{1-x^2}$

83. $\dfrac{3(1+x)^{1/3} - x(1+x)^{-2/3}}{(1+x)^{2/3}}$

84. $\dfrac{(7-3x)^{1/2} + \frac{3}{2}x(7-3x)^{-1/2}}{7-3x}$

85–90 ■ Rationalize Denominator Rationalize the denominator.

85. $\dfrac{1}{5-\sqrt{3}}$ **86.** $\dfrac{3}{2-\sqrt{5}}$

87. $\dfrac{2}{\sqrt{2}+\sqrt{7}}$ **88.** $\dfrac{1}{\sqrt{x}+1}$

89. $\dfrac{y}{\sqrt{3}+\sqrt{y}}$ **90.** $\dfrac{2(x-y)}{\sqrt{x}-\sqrt{y}}$

91–96 ■ Rationalize Numerator Rationalize the numerator.

91. $\dfrac{1-\sqrt{5}}{3}$ **92.** $\dfrac{\sqrt{3}+\sqrt{5}}{2}$

93. $\dfrac{\sqrt{r}+\sqrt{2}}{5}$ **94.** $\dfrac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}$

95. $\sqrt{x^2+1} - x$ **96.** $\sqrt{x+1} - \sqrt{x}$

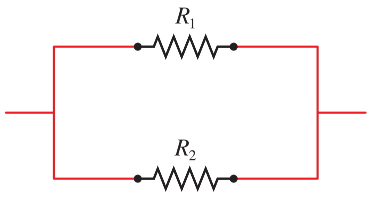
APPLICATIONS

97. Electrical Resistance If two electrical resistors with resistances R_1 and R_2 are connected in parallel (see the figure), then the total resistance R is given by

$$R = \dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}$$

(a) Simplify the expression for R.

(b) If $R_1 = 10$ ohms and $R_2 = 20$ ohms, what is the total resistance R?



98. Average Cost A clothing manufacturer finds that the cost of producing x shirts is $500 + 6x + 0.01x^2$ dollars.

(a) Explain why the average cost per shirt is given by the rational expression

$$A = \dfrac{500 + 6x + 0.01x^2}{x}$$

(b) Complete the table by calculating the average cost per shirt for the given values of x.

x	Average cost
10	
20	
50	
100	
200	
500	
1000	

DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

99. DISCOVER: Limiting Behavior of a Rational Expression The rational expression

$$\dfrac{x^2 - 9}{x - 3}$$

is not defined for $x = 3$. Complete the tables, and determine what value the expression approaches as x gets closer and closer to 3. Why is this reasonable? Factor the numerator of the expression and simplify to see why.

x	$\dfrac{x^2-9}{x-3}$
2.80	
2.90	
2.95	
2.99	
2.999	

x	$\dfrac{x^2-9}{x-3}$
3.20	
3.10	
3.05	
3.01	
3.001	

100. DISCUSS ■ WRITE: Is This Rationalization? In the expression $2/\sqrt{x}$ we would eliminate the radical if we were to square both numerator and denominator. Is this the same thing as rationalizing the denominator? Explain.