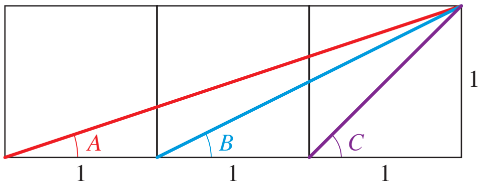
74. Find $\angle A + \angle B + \angle C$ in the figure. [*Hint:* First use an Addition Formula to find $\tan(A + B)$.]



APPLICATIONS

75. Adding an Echo A digital delay device echoes an input signal by repeating it a fixed length of time after it is received. If such a device receives the pure note $f_1(t) = 5 \sin t$ and echoes the pure note $f_2(t) = 5 \cos t$, then the combined sound is $f(t) = f_1(t) + f_2(t)$.

(a) Graph $y = f(t)$, and observe that the graph has the form of a sine curve $y = k \sin(t + \phi)$.

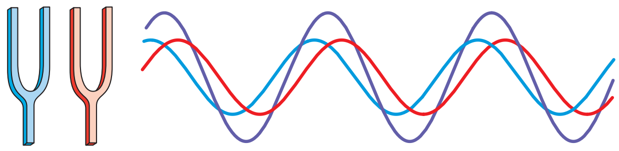
(b) Find k and ϕ.

76. Interference Two identical tuning forks are struck, one a fraction of a second after the other. The sounds produced are modeled by $f_1(t) = C \sin \omega t$ and $f_2(t) = C \sin(\omega t + \alpha)$. The two sound waves interfere to produce a single sound modeled by the sum of these functions

$$f(t) = C \sin \omega t + C \sin(\omega t + \alpha)$$

(a) Use the Addition Formula for Sine to show that f can be written in the form $f(t) = A \sin \omega t + B \cos \omega t$, where A and B are constants that depend on α.

(b) Suppose that $C = 10$ and $\alpha = \pi/3$. Find constants k and ϕ so that $f(t) = k \sin(\omega t + \phi)$.



DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

77. PROVE: Addition Formula for Sine In the text we proved only the Addition and Subtraction Formulas for Cosine. Use these formulas and the cofunction identities

$$\sin x = \cos\left(\frac{\pi}{2} - x\right)$$

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

to prove the Addition Formula for Sine. [*Hint:* To get started, use the first cofunction identity to write

$$\begin{aligned}\sin(s + t) &= \cos\left(\frac{\pi}{2} - (s + t)\right)\\ &= \cos\left(\left(\frac{\pi}{2} - s\right) - t\right)\end{aligned}$$

and use the Subtraction Formula for Cosine.]

78. PROVE: Addition Formula for Tangent Use the Addition Formulas for Cosine and Sine to prove the Addition Formula for Tangent. [*Hint:* Use

$$\tan(s + t) = \frac{\sin(s + t)}{\cos(s + t)}$$

and divide the numerator and denominator by $\cos s \cos t$.]

7.3 DOUBLE-ANGLE, HALF-ANGLE, AND PRODUCT-SUM FORMULAS

■ Double-Angle Formulas ■ Half-Angle Formulas ■ Evaluating Expressions Involving Inverse Trigonometric Functions ■ Product-Sum Formulas

The identities we consider in this section are consequences of the addition formulas. The **Double-Angle Formulas** allow us to find the values of the trigonometric functions at $2x$ from their values at x. The **Half-Angle Formulas** relate the values of the trigonometric functions at $\frac{1}{2}x$ to their values at x. The **Product-Sum Formulas** relate products of sines and cosines to sums of sines and cosines.

■ Double-Angle Formulas

The formulas in the box on the next page are immediate consequences of the addition formulas, which we proved in Section 7.2.

DOUBLE-ANGLE FORMULAS

Formula for sine: $\sin 2x = 2 \sin x \cos x$

Formulas for cosine:
$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \\ &= 2\cos^2 x - 1 \end{aligned}$$

Formula for tangent: $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$

The proofs for the formulas for cosine are given here. You are asked to prove the remaining formulas in Exercises 35 and 36.

Proof of Double-Angle Formulas for Cosine

$$\begin{aligned} \cos 2x &= \cos(x + x) \\ &= \cos x \cos x - \sin x \sin x \\ &= \cos^2 x - \sin^2 x \end{aligned}$$

The second and third formulas for $\cos 2x$ are obtained from the formula we just proved and the Pythagorean identity. Substituting $\cos^2 x = 1 - \sin^2 x$ gives

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= (1 - \sin^2 x) - \sin^2 x \\ &= 1 - 2\sin^2 x \end{aligned}$$

The third formula is obtained in the same way, by substituting $\sin^2 x = 1 - \cos^2 x$. ■

EXAMPLE 1 ■ Using the Double-Angle Formulas

If $\cos x = -\frac{2}{3}$ and x is in Quadrant II, find $\cos 2x$ and $\sin 2x$.

SOLUTION Using one of the Double-Angle Formulas for Cosine, we get

$$\begin{aligned} \cos 2x &= 2\cos^2 x - 1 \\ &= 2\left(-\frac{2}{3}\right)^2 - 1 = \frac{8}{9} - 1 = -\frac{1}{9} \end{aligned}$$

To use the formula $\sin 2x = 2 \sin x \cos x$, we need to find $\sin x$ first. We have

$$\sin x = \sqrt{1 - \cos^2 x} = \sqrt{1 - \left(-\tfrac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}$$

where we have used the positive square root because $\sin x$ is positive in Quadrant II. Thus

$$\begin{aligned} \sin 2x &= 2 \sin x \cos x \\ &= 2\left(\frac{\sqrt{5}}{3}\right)\left(-\frac{2}{3}\right) = -\frac{4\sqrt{5}}{9} \end{aligned}$$

Now Try Exercise 3 ■

EXAMPLE 2 ■ A Triple-Angle Formula

Write $\cos 3x$ in terms of $\cos x$.

SOLUTION

$$\begin{aligned}
\cos 3x &= \cos(2x + x) && \\
&= \cos 2x \cos x - \sin 2x \sin x && \text{Addition formula} \\
&= (2\cos^2 x - 1)\cos x - (2\sin x \cos x)\sin x && \text{Double-Angle Formulas} \\
&= 2\cos^3 x - \cos x - 2\sin^2 x \cos x && \text{Expand} \\
&= 2\cos^3 x - \cos x - 2\cos x\,(1 - \cos^2 x) && \text{Pythagorean identity} \\
&= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x && \text{Expand} \\
&= 4\cos^3 x - 3\cos x && \text{Simplify}
\end{aligned}$$

Now Try Exercise 109 ■

Example 2 shows that $\cos 3x$ can be written as a polynomial of degree 3 in $\cos x$. The identity $\cos 2x = 2\cos^2 x - 1$ shows that $\cos 2x$ is a polynomial of degree 2 in $\cos x$. In fact, for any natural number n we can write $\cos nx$ as a polynomial in $\cos x$ of degree n (see the note following Exercise 109). The analogous result for $\sin nx$ is not true in general.

EXAMPLE 3 ■ Proving an Identity

Prove the identity $\dfrac{\sin 3x}{\sin x \cos x} = 4\cos x - \sec x$.

SOLUTION We start with the left-hand side.

$$\begin{aligned}
\frac{\sin 3x}{\sin x \cos x} &= \frac{\sin(x + 2x)}{\sin x \cos x} && \\
&= \frac{\sin x \cos 2x + \cos x \sin 2x}{\sin x \cos x} && \text{Addition Formula} \\
&= \frac{\sin x\,(2\cos^2 x - 1) + \cos x\,(2\sin x \cos x)}{\sin x \cos x} && \text{Double-Angle Formulas} \\
&= \frac{\sin x\,(2\cos^2 x - 1)}{\sin x \cos x} + \frac{\cos x\,(2\sin x \cos x)}{\sin x \cos x} && \text{Separate fraction} \\
&= \frac{2\cos^2 x - 1}{\cos x} + 2\cos x && \text{Cancel} \\
&= 2\cos x - \frac{1}{\cos x} + 2\cos x && \text{Separate fraction} \\
&= 4\cos x - \sec x && \text{Reciprocal identity}
\end{aligned}$$

Now Try Exercise 87 ■

■ Half-Angle Formulas

The following formulas allow us to write any trigonometric expression involving even powers of sine and cosine in terms of the first power of cosine only. This technique is important in calculus. The Half-Angle Formulas are immediate consequences of these formulas.

FORMULAS FOR LOWERING POWERS

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Proof The first formula is obtained by solving for $\sin^2 x$ in the Double-Angle Formula $\cos 2x = 1 - 2\sin^2 x$. Similarly, the second formula is obtained by solving for $\cos^2 x$ in the Double-Angle Formula $\cos 2x = 2\cos^2 x - 1$.

The last formula follows from the first two and the reciprocal identities:

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{\dfrac{1 - \cos 2x}{2}}{\dfrac{1 + \cos 2x}{2}} = \frac{1 - \cos 2x}{1 + \cos 2x}$$

■

EXAMPLE 4 ■ Lowering Powers in a Trigonometric Expression

Express $\sin^2 x \cos^2 x$ in terms of the first power of cosine.

SOLUTION We use the formulas for lowering powers repeatedly.

$$\begin{aligned}
\sin^2 x \cos^2 x &= \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right) \\
&= \frac{1 - \cos^2 2x}{4} = \frac{1}{4} - \frac{1}{4}\cos^2 2x \\
&= \frac{1}{4} - \frac{1}{4}\left(\frac{1 + \cos 4x}{2}\right) = \frac{1}{4} - \frac{1}{8} - \frac{\cos 4x}{8} \\
&= \frac{1}{8} - \frac{1}{8}\cos 4x = \frac{1}{8}(1 - \cos 4x)
\end{aligned}$$

Another way to obtain this identity is to use the Double-Angle Formula for Sine in the form $\sin x \cos x = \frac{1}{2}\sin 2x$. Thus

$$\begin{aligned}
\sin^2 x \cos^2 x &= \frac{1}{4}\sin^2 2x = \frac{1}{4}\left(\frac{1 - \cos 4x}{2}\right) \\
&= \frac{1}{8}(1 - \cos 4x)
\end{aligned}$$

Now Try Exercise 11 ■

HALF-ANGLE FORMULAS

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1 - \cos u}{2}} \qquad \cos\frac{u}{2} = \pm\sqrt{\frac{1 + \cos u}{2}}$$

$$\tan\frac{u}{2} = \frac{1 - \cos u}{\sin u} = \frac{\sin u}{1 + \cos u}$$

The choice of the $+$ or $-$ sign depends on the quadrant in which $u/2$ lies.

Proof We substitute $x = u/2$ in the formulas for lowering powers and take the square root of each side. This gives the first two Half-Angle Formulas. In the case of the Half-Angle Formula for Tangent we get

$$\tan\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{1+\cos u}}$$

$$= \pm\sqrt{\left(\frac{1-\cos u}{1+\cos u}\right)\left(\frac{1-\cos u}{1-\cos u}\right)} \qquad \text{Multiply numerator and denominator by } 1-\cos u$$

$$= \pm\sqrt{\frac{(1-\cos u)^2}{1-\cos^2 u}} \qquad \text{Simplify}$$

$$= \pm\frac{|1-\cos u|}{|\sin u|} \qquad \sqrt{A^2} = |A| \text{ and } 1-\cos^2 u = \sin^2 u$$

Now, $1 - \cos u$ is nonnegative for all values of u. It is also true that $\sin u$ and $\tan(u/2)$ always have the same sign. (Verify this.) It follows that

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u}$$

The other Half-Angle Formula for Tangent is derived from this by multiplying the numerator and denominator by $1 + \cos u$. ■

EXAMPLE 5 ■ Using a Half-Angle Formula

Find the exact value of $\sin 22.5°$.

SOLUTION Since $22.5°$ is half of $45°$, we use the Half-Angle Formula for Sine with $u = 45°$. We choose the $+$ sign because $22.5°$ is in the first quadrant.

$$\sin\frac{45°}{2} = \sqrt{\frac{1-\cos 45°}{2}} \qquad \text{Half-Angle Formula}$$

$$= \sqrt{\frac{1-\sqrt{2}/2}{2}} \qquad \cos 45° = \sqrt{2}/2$$

$$= \sqrt{\frac{2-\sqrt{2}}{4}} \qquad \text{Common denominator}$$

$$= \tfrac{1}{2}\sqrt{2-\sqrt{2}} \qquad \text{Simplify}$$

Now Try Exercise 17 ■

EXAMPLE 6 ■ Using a Half-Angle Formula

Find $\tan(u/2)$ if $\sin u = \frac{2}{5}$ and u is in Quadrant II.

SOLUTION To use the Half-Angle Formula for Tangent, we first need to find $\cos u$. Since cosine is negative in Quadrant II, we have

$$\cos u = -\sqrt{1-\sin^2 u}$$

$$= -\sqrt{1-\left(\tfrac{2}{5}\right)^2} = -\frac{\sqrt{21}}{5}$$

Thus

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u}$$

$$= \frac{1+\sqrt{21}/5}{\frac{2}{5}} = \frac{5+\sqrt{21}}{2}$$

Now Try Exercise 37 ■

Evaluating Expressions Involving Inverse Trigonometric Functions

Expressions involving trigonometric functions and their inverses arise in calculus. In the next examples we illustrate how to evaluate such expressions.

EXAMPLE 7 ■ Simplifying an Expression Involving an Inverse Trigonometric Function

Write $\sin(2\cos^{-1}x)$ as an algebraic expression in x only, where $-1 \le x \le 1$.

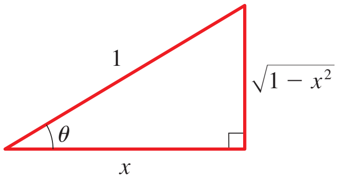


FIGURE 1

SOLUTION Let $\theta = \cos^{-1}x$, and sketch a triangle as in Figure 1. We need to find $\sin 2\theta$, but from the triangle we can find trigonometric functions of θ only, not 2θ. So we use the Double-Angle Formula for Sine.

$$
\begin{aligned}
\sin(2\cos^{-1}x) &= \sin 2\theta && \cos^{-1}x = \theta \\
&= 2\sin\theta\cos\theta && \text{Double-Angle Formula} \\
&= 2x\sqrt{1-x^2} && \text{From the triangle}
\end{aligned}
$$

Now Try Exercises 43 and 47 ■

EXAMPLE 8 ■ Evaluating an Expression Involving Trigonometric Functions

Evaluate $\sin 2\theta$, where $\cos\theta = -\frac{2}{5}$ with θ in Quadrant II.

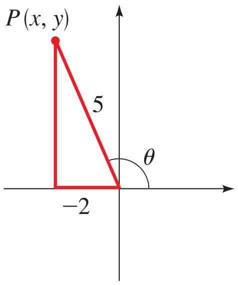


FIGURE 2

SOLUTION We first sketch the angle θ in standard position with terminal side in Quadrant II as in Figure 2. Since $\cos\theta = x/r = -\frac{2}{5}$, we can label a side and the hypotenuse of the triangle in Figure 2. To find the remaining side, we use the Pythagorean Theorem.

$$
\begin{aligned}
x^2 + y^2 &= r^2 && \text{Pythagorean Theorem} \\
(-2)^2 + y^2 &= 5^2 && x = -2, \quad r = 5 \\
y &= \pm\sqrt{21} && \text{Solve for } y^2 \\
y &= +\sqrt{21} && \text{Because } y > 0
\end{aligned}
$$

We can now use the Double-Angle Formula for Sine.

$$
\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta && \text{Double-Angle Formula} \\
&= 2\left(\frac{\sqrt{21}}{5}\right)\left(-\frac{2}{5}\right) && \text{From the triangle} \\
&= -\frac{4\sqrt{21}}{25} && \text{Simplify}
\end{aligned}
$$

Now Try Exercise 51 ■

DISCOVERY PROJECT

Where to Sit at the Movies

To best view a painting or a movie requires that the viewing angle be as large as possible. If the painting or movie screen is at a height above eye level, then being too far away or too close results in a small viewing angle and hence a poor viewing experience. So what is the best distance from which to view a movie or a painting? In this project we use trigonometry to find the best location from which to view a painting or a movie. You can find the project at **www.stewartmath.com**.

Product-Sum Formulas

It is possible to write the product $\sin u \cos v$ as a sum of trigonometric functions. To see this, consider the Addition and Subtraction Formulas for Sine:

$$\sin(u + v) = \sin u \cos v + \cos u \sin v$$

$$\sin(u - v) = \sin u \cos v - \cos u \sin v$$

Adding the left- and right-hand sides of these formulas gives

$$\sin(u + v) + \sin(u - v) = 2 \sin u \cos v$$

Dividing by 2 gives the formula

$$\sin u \cos v = \tfrac{1}{2}[\sin(u + v) + \sin(u - v)]$$

The other three **Product-to-Sum Formulas** follow from the Addition Formulas in a similar way.

PRODUCT-TO-SUM FORMULAS

$$\sin u \cos v = \tfrac{1}{2}[\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \tfrac{1}{2}[\sin(u + v) - \sin(u - v)]$$

$$\cos u \cos v = \tfrac{1}{2}[\cos(u + v) + \cos(u - v)]$$

$$\sin u \sin v = \tfrac{1}{2}[\cos(u - v) - \cos(u + v)]$$

EXAMPLE 9 ■ Expressing a Trigonometric Product as a Sum

Express $\sin 3x \sin 5x$ as a sum of trigonometric functions.

SOLUTION Using the fourth Product-to-Sum Formula with $u = 3x$ and $v = 5x$ and the fact that cosine is an even function, we get

$$\begin{aligned} \sin 3x \sin 5x &= \tfrac{1}{2}[\cos(3x - 5x) - \cos(3x + 5x)] \\ &= \tfrac{1}{2}\cos(-2x) - \tfrac{1}{2}\cos 8x \\ &= \tfrac{1}{2}\cos 2x - \tfrac{1}{2}\cos 8x \end{aligned}$$

Now Try Exercise 55 ■

The Product-to-Sum Formulas can also be used as Sum-to-Product Formulas. This is possible because the right-hand side of each Product-to-Sum Formula is a sum and the left side is a product. For example, if we let

$$u = \frac{x + y}{2} \qquad \text{and} \qquad v = \frac{x - y}{2}$$

in the first Product-to-Sum Formula, we get

$$\sin \frac{x + y}{2} \cos \frac{x - y}{2} = \tfrac{1}{2}(\sin x + \sin y)$$

so

$$\sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}$$

The remaining three of the following **Sum-to-Product Formulas** are obtained in a similar manner.

SUM-TO-PRODUCT FORMULAS

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

EXAMPLE 10 ■ Expressing a Trigonometric Sum as a Product

Write $\sin 7x + \sin 3x$ as a product.

SOLUTION The first Sum-to-Product Formula gives

$$\sin 7x + \sin 3x = 2\sin\frac{7x+3x}{2}\cos\frac{7x-3x}{2}$$

$$= 2\sin 5x\cos 2x$$

Now Try Exercise 61 ■

EXAMPLE 11 ■ Proving an Identity

Verify the identity $\dfrac{\sin 3x - \sin x}{\cos 3x + \cos x} = \tan x$.

SOLUTION We apply the second Sum-to-Product Formula to the numerator and the third formula to the denominator.

$$\text{LHS} = \frac{\sin 3x - \sin x}{\cos 3x + \cos x} = \frac{2\cos\frac{3x+x}{2}\sin\frac{3x-x}{2}}{2\cos\frac{3x+x}{2}\cos\frac{3x-x}{2}} \quad \text{Sum-to-Product Formulas}$$

$$= \frac{2\cos 2x\sin x}{2\cos 2x\cos x} \quad \text{Simplify}$$

$$= \frac{\sin x}{\cos x} = \tan x = \text{RHS} \quad \text{Cancel}$$

Now Try Exercise 93 ■

7.3 EXERCISES

CONCEPTS

1. If we know the values of $\sin x$ and $\cos x$, we can find the value of $\sin 2x$ by using the ________ Formula for Sine. State the formula: $\sin 2x =$ ________________.

2. If we know the value of $\cos x$ and the quadrant in which $x/2$ lies, we can find the value of $\sin(x/2)$ by using the ________ Formula for Sine. State the formula: $\sin(x/2) =$ ________________.

SKILLS

3–10 ■ Double Angle Formulas Find $\sin 2x$, $\cos 2x$, and $\tan 2x$ from the given information.

3. $\sin x = \frac{5}{13}$, x in Quadrant I

4. $\tan x = -\frac{4}{3}$, x in Quadrant II

5. $\cos x = \frac{4}{5}$, $\csc x < 0$

6. $\csc x = 4$, $\tan x < 0$

7. $\sin x = -\frac{3}{5}$, x in Quadrant III

8. $\sec x = 2$, x in Quadrant IV

9. $\tan x = -\frac{1}{3}$, $\cos x > 0$

10. $\cot x = \frac{2}{3}$, $\sin x > 0$

11–16 ■ Lowering Powers in a Trigonometric Expression Use the formulas for lowering powers to rewrite the expression in terms of the first power of cosine, as in Example 4.

11. $\sin^4 x$

12. $\cos^4 x$

13. $\cos^2 x \sin^4 x$

14. $\cos^4 x \sin^2 x$

15. $\cos^4 x \sin^4 x$

16. $\cos^6 x$

17–28 ■ Half Angle Formulas Use an appropriate Half-Angle Formula to find the exact value of the expression.

17. $\sin 15°$

18. $\tan 15°$

19. $\tan 22.5°$

20. $\sin 75°$

21. $\cos 165°$

22. $\cos 112.5°$

23. $\tan \dfrac{\pi}{8}$

24. $\cos \dfrac{3\pi}{8}$

25. $\cos \dfrac{\pi}{12}$

26. $\tan \dfrac{5\pi}{12}$

27. $\sin \dfrac{9\pi}{8}$

28. $\sin \dfrac{11\pi}{12}$

29–34 ■ Double- and Half-Angle Formulas Simplify the expression by using a Double-Angle Formula or a Half-Angle Formula.

29. (a) $2 \sin 18° \cos 18°$ **(b)** $2 \sin 3\theta \cos 3\theta$

30. (a) $\dfrac{2 \tan 7°}{1 - \tan^2 7°}$ **(b)** $\dfrac{2 \tan 7\theta}{1 - \tan^2 7\theta}$

31. (a) $\cos^2 34° - \sin^2 34°$ **(b)** $\cos^2 5\theta - \sin^2 5\theta$

32. (a) $\cos^2 \dfrac{\theta}{2} - \sin^2 \dfrac{\theta}{2}$ **(b)** $2 \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2}$

33. (a) $\dfrac{\sin 8°}{1 + \cos 8°}$ **(b)** $\dfrac{1 - \cos 4\theta}{\sin 4\theta}$

34. (a) $\sqrt{\dfrac{1 - \cos 30°}{2}}$ **(b)** $\sqrt{\dfrac{1 - \cos 8\theta}{2}}$

35. Proving a Double-Angle Formula Use the Addition Formula for Sine to prove the Double-Angle Formula for Sine.

36. Proving a Double-Angle Formula Use the Addition Formula for Tangent to prove the Double-Angle Formula for Tangent.

37–42 ■ Using a Half-Angle Formula Find $\sin \dfrac{x}{2}$, $\cos \dfrac{x}{2}$, and $\tan \dfrac{x}{2}$ from the given information.

37. $\sin x = \frac{3}{5}$, $0° < x < 90°$

38. $\cos x = -\frac{4}{5}$, $180° < x < 270°$

39. $\csc x = 3$, $90° < x < 180°$

40. $\tan x = 1$, $0° < x < 90°$

41. $\sec x = \frac{3}{2}$, $270° < x < 360°$

42. $\cot x = 5$, $180° < x < 270°$

43–46 ■ Expressions Involving Inverse Trigonometric Functions Write the given expression as an algebraic expression in x.

43. $\sin(2 \tan^{-1} x)$

44. $\tan(2 \cos^{-1} x)$

45. $\sin\left(\frac{1}{2} \cos^{-1} x\right)$

46. $\cos(2 \sin^{-1} x)$

47–50 ■ Expressions Involving Inverse Trigonometric Functions Find the exact value of the given expression.

47. $\sin\left(2 \cos^{-1} \frac{7}{25}\right)$

48. $\cos\left(2 \tan^{-1} \frac{12}{5}\right)$

49. $\sec\left(2 \sin^{-1} \frac{1}{4}\right)$

50. $\tan\left(\frac{1}{2} \cos^{-1} \frac{2}{3}\right)$

51–54 ■ Evaluating an Expression Involving Trigonometric Functions Evaluate each expression under the given conditions.

51. $\cos 2\theta$; $\sin \theta = -\frac{3}{5}$, θ in Quadrant III

52. $\sin(\theta/2)$; $\tan \theta = -\frac{5}{12}$, θ in Quadrant IV

53. $\sin 2\theta$; $\sin \theta = \frac{1}{7}$, θ in Quadrant II

54. $\tan 2\theta$; $\cos \theta = \frac{3}{5}$, θ in Quadrant I

55–60 ■ Product-to-Sum Formulas Write the product as a sum.

55. $\sin 2x \cos 3x$

56. $\sin x \sin 5x$

57. $\cos x \sin 4x$

58. $\cos 5x \cos 3x$

59. $3 \cos 4x \cos 7x$

60. $11 \sin \dfrac{x}{2} \cos \dfrac{x}{4}$

61–66 ■ Sum-to-Product Formulas Write the sum as a product.

61. $\sin 5x + \sin 3x$

62. $\sin x - \sin 4x$

63. $\cos 4x - \cos 6x$

64. $\cos 9x + \cos 2x$

65. $\sin 2x - \sin 7x$

66. $\sin 3x + \sin 4x$

67–72 ■ Value of a Product or Sum Find the value of the product or sum.

67. $2 \sin 52.5° \sin 97.5°$

68. $3 \cos 37.5° \cos 7.5°$

69. $\cos 37.5° \sin 7.5°$

70. $\sin 75° + \sin 15°$

71. $\cos 255° - \cos 195°$

72. $\cos \dfrac{\pi}{12} + \cos \dfrac{5\pi}{12}$

73–92 ■ Proving Identities Prove the identity.

73. $\cos^2 5x - \sin^2 5x = \cos 10x$

74. $\sin 8x = 2 \sin 4x \cos 4x$

75. $(\sin x + \cos x)^2 = 1 + \sin 2x$

76. $\cos^4 x - \sin^4 x = \cos 2x$

77. $\dfrac{2\tan x}{1+\tan^2 x} = \sin 2x$

78. $\dfrac{1-\cos 2x}{\sin 2x} = \tan x$

79. $\tan\left(\dfrac{x}{2}\right) + \cos x \tan\left(\dfrac{x}{2}\right) = \sin x$

80. $\tan\left(\dfrac{x}{2}\right) + \csc x = \dfrac{2-\cos x}{\sin x}$

81. $\dfrac{\sin 4x}{\sin x} = 4\cos x \cos 2x$

82. $\dfrac{1+\sin 2x}{\sin 2x} = 1 + \frac{1}{2}\sec x \csc x$

83. $\dfrac{2(\tan x - \cot x)}{\tan^2 x - \cot^2 x} = \sin 2x$

84. $\tan x = \dfrac{\sin 2x}{1+\cos 2x}$

85. $\cot 2x = \dfrac{1-\tan^2 x}{2\tan x}$

86. $4(\sin^6 x + \cos^6 x) = 4 - 3\sin^2 2x$

87. $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

88. $\dfrac{\sin 3x + \cos 3x}{\cos x - \sin x} = 1 + 4\sin x \cos x$

89. $\dfrac{\sin x + \sin 5x}{\cos x + \cos 5x} = \tan 3x$

90. $\dfrac{\sin 3x + \sin 7x}{\cos 3x - \cos 7x} = \cot 2x$

91. $\dfrac{\sin 10x}{\sin 9x + \sin x} = \dfrac{\cos 5x}{\cos 4x}$

92. $\dfrac{\sin x + \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x} = \tan 3x$

93. $\dfrac{\sin x + \sin y}{\cos x + \cos y} = \tan\left(\dfrac{x+y}{2}\right)$

94. $\tan y = \dfrac{\sin(x+y) - \sin(x-y)}{\cos(x+y) + \cos(x-y)}$

95. $\tan^2\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right) = \dfrac{1+\sin x}{1-\sin x}$

96. $(1 - \cos 4x)(2 + \tan^2 x + \cot^2 x) = 8$

97–100 ■ Sum-to-Product Formulas Use a Sum-to-Product Formula to show the following.

97. $\sin 130° - \sin 110° = -\sin 10°$

98. $\cos 100° - \cos 200° = \sin 50°$

99. $\sin 45° + \sin 15° = \sin 75°$

100. $\cos 87° + \cos 33° = \sin 63°$

SKILLS Plus

101. Proving an Identity Prove the identity

$$\frac{\sin x + \sin 2x + \sin 3x + \sin 4x + \sin 5x}{\cos x + \cos 2x + \cos 3x + \cos 4x + \cos 5x} = \tan 3x$$

102. Proving an Identity Use the identity

$$\sin 2x = 2\sin x \cos x$$

n times to show that

$$\sin(2^n x) = 2^n \sin x \cos x \cos 2x \cos 4x \cdots \cos 2^{n-1} x$$

103–104 ■ Identities Involving Inverse Trigonometric Functions Prove the identity.

103. $2\sin^{-1} x = \cos^{-1}(1 - 2x^2)$, $0 \le x \le 1$ [*Hint:* Let $u = \sin^{-1} x$, so that $x = \sin u$. Use a Double-Angle Formula to show that $1 - 2x^2 = \cos 2u$.]

104. $2\tan^{-1}\left(\dfrac{1}{x}\right) = \cos^{-1}\left(\dfrac{x^2-1}{x^2+1}\right)$

[*Hint:* Let $u = \tan^{-1}\left(\dfrac{1}{x}\right)$, so that $x = \dfrac{1}{\tan u} = \cot u$. Use a Double-Angle Formula to show that $\dfrac{x^2-1}{x^2+1} = \dfrac{\cot^2 u - 1}{\csc^2 u} = \cos 2u$.]

105–107 ■ Discovering an Identity Graphically In these problems we discover an identity graphically and then prove the identity.

105. **(a)** Graph $f(x) = \dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x}$, and make a conjecture.

(b) Prove the conjecture you made in part (a).

106. **(a)** Graph $f(x) = \cos 2x + 2\sin^2 x$, and make a conjecture.

(b) Prove the conjecture you made in part (a).

107. Let $f(x) = \sin 6x + \sin 7x$.

(a) Graph $y = f(x)$.

(b) Verify that $f(x) = 2\cos\frac{1}{2}x \sin\frac{13}{2}x$.

(c) Graph $y = 2\cos\frac{1}{2}x$ and $y = -2\cos\frac{1}{2}x$, together with the graph in part (a), in the same viewing rectangle. How are these graphs related to the graph of f?

108. A Cubic Equation Let $3x = \pi/3$, and let $y = \cos x$. Use the result of Example 2 to show that y satisfies the equation

$$8y^3 - 6y - 1 = 0$$

[*Note:* This equation has roots of a certain kind that are used to show that the angle $\pi/3$ cannot be trisected by using a ruler and compass only.]

109. Tchebycheff Polynomials

(a) Show that there is a polynomial $P(t)$ of degree 4 such that $\cos 4x = P(\cos x)$ (see Example 2).

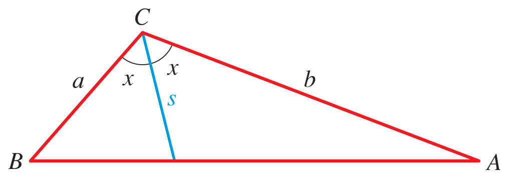
(b) Show that there is a polynomial $Q(t)$ of degree 5 such that $\cos 5x = Q(\cos x)$.

[*Note:* In general, there is a polynomial $P_n(t)$ of degree n such that $\cos nx = P_n(\cos x)$. These polynomials are called *Tchebycheff polynomials*, after the Russian mathematician P. L. Tchebycheff (1821–1894).]

110. Length of a Bisector In triangle ABC (see the figure) the line segment s bisects angle C. Show that the length of s is given by

$$s = \frac{2ab \cos x}{a + b}$$

[*Hint:* Use the Law of Sines.]



111. Angles of a Triangle If A, B, and C are the angles in a triangle, show that

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$$

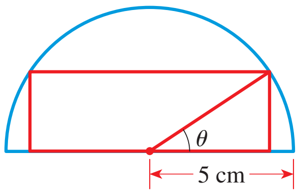
112. Largest Area A rectangle is to be inscribed in a semicircle of radius 5 cm as shown in the following figure.

(a) Show that the area of the rectangle is modeled by the function

$$A(\theta) = 25 \sin 2\theta$$

(b) Find the largest possible area for such an inscribed rectangle. [*Hint:* Use the fact that $\sin u$ achieves its maximum value at $u = \pi/2$.]

(c) Find the dimensions of the inscribed rectangle with the largest possible area.



APPLICATIONS

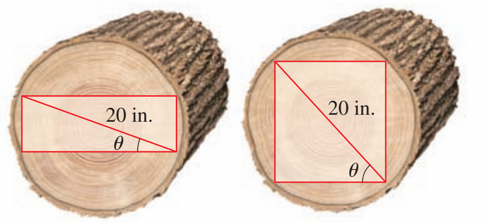
113. Sawing a Wooden Beam A rectangular beam is to be cut from a cylindrical log of diameter 20 in.

(a) Show that the cross-sectional area of the beam is modeled by the function

$$A(\theta) = 200 \sin 2\theta$$

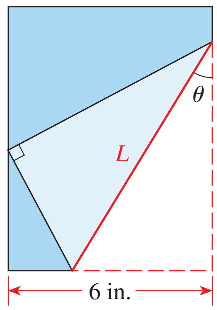
where θ is as shown in the figure.

(b) Show that the maximum cross-sectional area of such a beam is 200 in^2. [*Hint:* Use the fact that $\sin u$ achieves its maximum value at $u = \pi/2$.]



114. Length of a Fold The lower right-hand corner of a long piece of paper 6 in. wide is folded over to the left-hand edge as shown. The length L of the fold depends on the angle θ. Show that

$$L = \frac{3}{\sin\theta \cos^2\theta}$$



115. Sound Beats When two pure notes that are close in frequency are played together, their sounds interfere to produce *beats*; that is, the loudness (or amplitude) of the sound alternately increases and decreases. If the two notes are given by

$$f_1(t) = \cos 11t \qquad \text{and} \qquad f_2(t) = \cos 13t$$

the resulting sound is $f(t) = f_1(t) + f_2(t)$.

(a) Graph the function $y = f(t)$.

(b) Verify that $f(t) = 2 \cos t \cos 12t$.

(c) Graph $y = 2 \cos t$ and $y = -2 \cos t$, together with the graph in part (a), in the same viewing rectangle. How do these graphs describe the variation in the loudness of the sound?

116. Touch-Tone Telephones When a key is pressed on a touch-tone telephone, the keypad generates two pure tones, which combine to produce a sound that uniquely identifies the key. The figure shows the low frequency f_1 and the high frequency f_2 associated with each key. Pressing a key produces the sound wave $y = \sin(2\pi f_1 t) + \sin(2\pi f_2 t)$.

(a) Find the function that models the sound produced when the 4 key is pressed.

(b) Use a Sum-to-Product Formula to express the sound generated by the 4 key as a product of a sine and a cosine function.

(c) Graph the sound wave generated by the 4 key from $t = 0$ to $t = 0.006$ s.