

appear identical in the viewing rectangle. Does this prove that the equation $f(x) = g(x)$ is an identity? Explain.

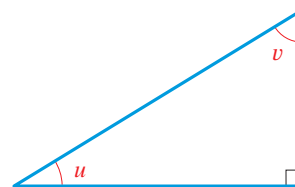
- 117. DISCOVER: Making Up Your Own Identity** If you start with a trigonometric expression and rewrite it or simplify it, then setting the original expression equal to the rewritten expression yields a trigonometric identity. For instance, from Example 1 we get the identity

$$\cos t + \tan t \sin t = \sec t$$

Use this technique to make up your own identity, then give it to a classmate to verify.

- 118. DISCUSS: Cofunction Identities** In the right triangle shown, explain why $v = (\pi/2) - u$. Explain how you can

obtain all six cofunction identities from this triangle for $0 < u < \pi/2$.



Note that u and v are complementary angles. So the cofunction identities state that “a trigonometric function of an angle u is equal to the corresponding cofunction of the complementary angle v .”

7.2 ADDITION AND SUBTRACTION FORMULAS

■ Addition and Subtraction Formulas ■ Evaluating Expressions Involving Inverse Trigonometric Functions ■ Expressions of the form $A \sin x + B \cos x$

■ Addition and Subtraction Formulas

We now derive identities for trigonometric functions of sums and differences.

ADDITION AND SUBTRACTION FORMULAS

Formulas for sine: $\sin(s + t) = \sin s \cos t + \cos s \sin t$
 $\sin(s - t) = \sin s \cos t - \cos s \sin t$

Formulas for cosine: $\cos(s + t) = \cos s \cos t - \sin s \sin t$
 $\cos(s - t) = \cos s \cos t + \sin s \sin t$

Formulas for tangent: $\tan(s + t) = \frac{\tan s + \tan t}{1 - \tan s \tan t}$
 $\tan(s - t) = \frac{\tan s - \tan t}{1 + \tan s \tan t}$

Proof of Addition Formula for Cosine To prove the formula

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

we use Figure 1. In the figure, the distances t , $s + t$, and $-s$ have been marked on the unit circle, starting at $P_0(1, 0)$ and terminating at Q_1 , P_1 , and Q_0 , respectively. The coordinates of these points are as follows:

$P_0(1, 0)$	$Q_0(\cos(-s), \sin(-s))$
$P_1(\cos(s + t), \sin(s + t))$	$Q_1(\cos t, \sin t)$

Since $\cos(-s) = \cos s$ and $\sin(-s) = -\sin s$, it follows that the point Q_0 has the coordinates $Q_0(\cos s, -\sin s)$. Notice that the distances between P_0 and P_1 and between Q_0 and Q_1 measured along the arc of the circle are equal. Since equal arcs are subtended by equal chords, it follows that $d(P_0, P_1) = d(Q_0, Q_1)$. Using the Distance Formula, we get

$$\sqrt{[\cos(s + t) - 1]^2 + [\sin(s + t) - 0]^2} = \sqrt{(\cos t - \cos s)^2 + (\sin t + \sin s)^2}$$

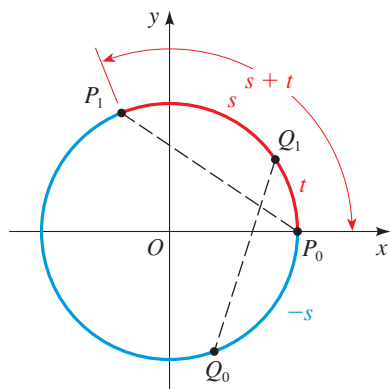


FIGURE 1



JEAN BAPTISTE JOSEPH FOURIER (1768–1830) is responsible for the most powerful application of the trigonometric functions (see the margin note on page 427). He used sums of these functions to describe such physical phenomena as the transmission of sound and the flow of heat.

Orphaned as a young boy, Fourier was educated in a military school, where he became a mathematics teacher at the age of 20. He was later appointed professor at the École Polytechnique but resigned this position to accompany Napoleon on his expedition to Egypt, where Fourier served as governor. After returning to France, he began conducting experiments on heat. The French Academy refused to publish his early papers on this subject because of his lack of rigor. Fourier eventually became Secretary of the Academy and in this capacity had his papers published in their original form. Probably because of his study of heat and his years in the deserts of Egypt, Fourier became obsessed with keeping himself warm—he wore several layers of clothes, even in the summer, and kept his rooms at unbearably high temperatures. Evidently, these habits overburdened his heart and contributed to his death at the age of 62.

Squaring both sides and expanding, we have

$$\begin{aligned} \cos^2(s+t) - 2\cos(s+t) + 1 + \sin^2(s+t) \\ = \cos^2 t - 2\cos s \cos t + \cos^2 s + \sin^2 t + 2\sin s \sin t + \sin^2 s \end{aligned}$$

These add to 1
These add to 1
These add to 1

Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ three times gives

$$2 - 2\cos(s+t) = 2 - 2\cos s \cos t + 2\sin s \sin t$$

Finally, subtracting 2 from each side and dividing both sides by -2 , we get

$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$

which proves the Addition Formula for Cosine. ■

Proof of Subtraction Formula for Cosine Replacing t with $-t$ in the Addition Formula for Cosine, we get

$$\begin{aligned} \cos(s-t) &= \cos(s+(-t)) \\ &= \cos s \cos(-t) - \sin s \sin(-t) && \text{Addition Formula for Cosine} \\ &= \cos s \cos t + \sin s \sin t && \text{Even-odd identities} \end{aligned}$$

This proves the Subtraction Formula for Cosine. ■

See Exercises 77 and 78 for proofs of the other Addition Formulas.

EXAMPLE 1 ■ Using the Addition and Subtraction Formulas

Find the exact value of each expression.

(a) $\cos 75^\circ$ (b) $\cos \frac{\pi}{12}$

SOLUTION

(a) Notice that $75^\circ = 45^\circ + 30^\circ$. Since we know the exact values of sine and cosine at 45° and 30° , we use the Addition Formula for Cosine to get

$$\begin{aligned} \cos 75^\circ &= \cos(45^\circ + 30^\circ) \\ &= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ \\ &= \frac{\sqrt{2}}{2} \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \frac{1}{2} = \frac{\sqrt{2}\sqrt{3} - \sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4} \end{aligned}$$

(b) Since $\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$, the Subtraction Formula for Cosine gives

$$\begin{aligned} \cos \frac{\pi}{12} &= \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\ &= \cos \frac{\pi}{4} \cos \frac{\pi}{6} + \sin \frac{\pi}{4} \sin \frac{\pi}{6} \\ &= \frac{\sqrt{2}}{2} \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4} \end{aligned}$$

Now Try Exercises 3 and 9 ■

EXAMPLE 2 ■ Using the Addition Formula for Sine

Find the exact value of the expression $\sin 20^\circ \cos 40^\circ + \cos 20^\circ \sin 40^\circ$.

SOLUTION We recognize the expression as the right-hand side of the Addition Formula for Sine with $s = 20^\circ$ and $t = 40^\circ$. So we have

$$\sin 20^\circ \cos 40^\circ + \cos 20^\circ \sin 40^\circ = \sin(20^\circ + 40^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

 **Now Try Exercise 15**

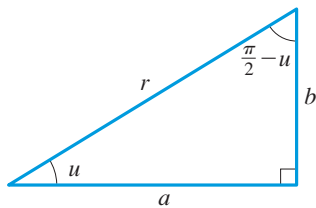
EXAMPLE 3 ■ Proving a Cofunction Identity

Prove the cofunction identity $\cos\left(\frac{\pi}{2} - u\right) = \sin u$.

SOLUTION By the Subtraction Formula for Cosine we have

$$\begin{aligned}\cos\left(\frac{\pi}{2} - u\right) &= \cos \frac{\pi}{2} \cos u + \sin \frac{\pi}{2} \sin u \\ &= 0 \cdot \cos u + 1 \cdot \sin u = \sin u\end{aligned}$$

 **Now Try Exercises 21 and 25**



$$\cos\left(\frac{\pi}{2} - u\right) = \frac{b}{r} = \sin u$$

For acute angles, the cofunction identity in Example 3, as well as the other cofunction identities, can also be derived from the figure in the margin.

EXAMPLE 4 ■ Proving an Identity

Verify the identity $\frac{1 + \tan x}{1 - \tan x} = \tan\left(\frac{\pi}{4} + x\right)$.

SOLUTION Starting with the right-hand side and using the Addition Formula for Tangent, we get

$$\begin{aligned}\text{RHS} &= \tan\left(\frac{\pi}{4} + x\right) = \frac{\tan \frac{\pi}{4} + \tan x}{1 - \tan \frac{\pi}{4} \tan x} \\ &= \frac{1 + \tan x}{1 - \tan x} = \text{LHS}\end{aligned}$$

 **Now Try Exercise 33**

The next example is a typical use of the Addition and Subtraction Formulas in calculus.

EXAMPLE 5 ■ An Identity from Calculus

If $f(x) = \sin x$, show that

$$\frac{f(x+h) - f(x)}{h} = \sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right)$$

SOLUTION

$$\begin{aligned}
 \frac{f(x+h) - f(x)}{h} &= \frac{\sin(x+h) - \sin x}{h} && \text{Definition of } f \\
 &= \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} && \text{Addition Formula for Sine} \\
 &= \frac{\sin x (\cos h - 1) + \cos x \sin h}{h} && \text{Factor} \\
 &= \sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) && \text{Separate the fraction}
 \end{aligned}$$

 Now Try Exercise 65

■ Evaluating Expressions Involving Inverse Trigonometric Functions

Expressions involving trigonometric functions and their inverses arise in calculus. In the next examples we illustrate how to evaluate such expressions.

EXAMPLE 6 ■ Simplifying an Expression Involving Inverse Trigonometric Functions

Write $\sin(\cos^{-1}x + \tan^{-1}y)$ as an algebraic expression in x and y , where $-1 \leq x \leq 1$ and y is any real number.

SOLUTION Let $\theta = \cos^{-1}x$ and $\phi = \tan^{-1}y$. Using the methods of Section 6.4, we sketch triangles with angles θ and ϕ such that $\cos \theta = x$ and $\tan \phi = y$ (see Figure 2). From the triangles we have

$$\sin \theta = \sqrt{1-x^2} \quad \cos \phi = \frac{1}{\sqrt{1+y^2}} \quad \sin \phi = \frac{y}{\sqrt{1+y^2}}$$

From the Addition Formula for Sine we have

$$\begin{aligned}
 \sin(\cos^{-1}x + \tan^{-1}y) &= \sin(\theta + \phi) \\
 &= \sin \theta \cos \phi + \cos \theta \sin \phi && \text{Addition Formula for Sine} \\
 &= \sqrt{1-x^2} \frac{1}{\sqrt{1+y^2}} + x \frac{y}{\sqrt{1+y^2}} && \text{From triangles} \\
 &= \frac{1}{\sqrt{1+y^2}} (\sqrt{1-x^2} + xy) && \text{Factor } \frac{1}{\sqrt{1+y^2}}
 \end{aligned}$$

 Now Try Exercises 47 and 51

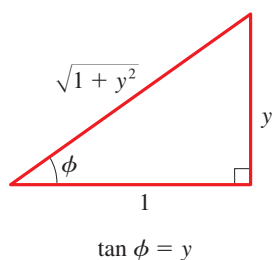
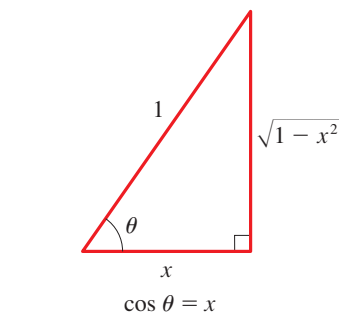


FIGURE 2

EXAMPLE 7 ■ Evaluating an Expression Involving Trigonometric Functions

Evaluate $\sin(\theta + \phi)$, where $\sin \theta = \frac{12}{13}$ with θ in Quadrant II and $\tan \phi = \frac{3}{4}$ with ϕ in Quadrant III.

SOLUTION We first sketch the angles θ and ϕ in standard position with terminal sides in the appropriate quadrants as in Figure 3. Since $\sin \theta = y/r = \frac{12}{13}$, we can label a side

and the hypotenuse in the triangle in Figure 3(a). To find the remaining side, we use the Pythagorean Theorem.

$$x^2 + y^2 = r^2 \quad \text{Pythagorean Theorem}$$

$$x^2 + 12^2 = 13^2 \quad y = 12, \quad r = 13$$

$$x^2 = 25 \quad \text{Solve for } x^2$$

$$x = -5 \quad \text{Because } x < 0$$

Similarly, since $\tan \phi = y/x = \frac{3}{4}$, we can label two sides of the triangle in Figure 3(b) and then use the Pythagorean Theorem to find the hypotenuse.

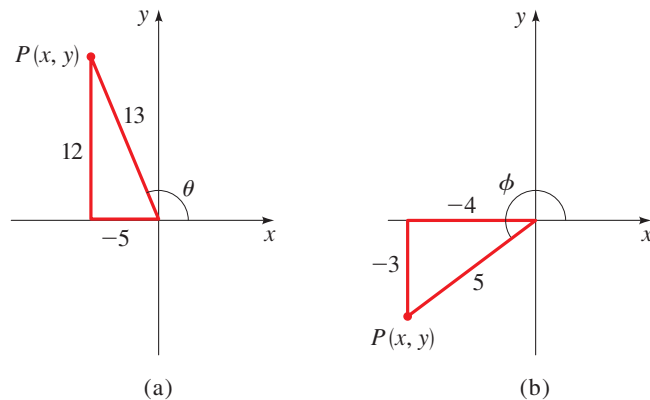


FIGURE 3

Now, to find $\sin(\theta + \phi)$, we use the Addition Formula for Sine and the triangles in Figure 3.

$$\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi \quad \text{Addition Formula}$$

$$= \left(\frac{12}{13}\right)\left(-\frac{4}{5}\right) + \left(-\frac{5}{13}\right)\left(-\frac{3}{5}\right) \quad \text{From triangles}$$

$$= -\frac{33}{65} \quad \text{Calculate}$$

Now Try Exercise 55

Expressions of the Form $A \sin x + B \cos x$

We can write expressions of the form $A \sin x + B \cos x$ in terms of a single trigonometric function using the Addition Formula for Sine. For example, consider the expression

$$\frac{1}{2} \sin x + \frac{\sqrt{3}}{2} \cos x$$

If we set $\phi = \pi/3$, then $\cos \phi = \frac{1}{2}$ and $\sin \phi = \sqrt{3}/2$, and we can write

$$\begin{aligned} \frac{1}{2} \sin x + \frac{\sqrt{3}}{2} \cos x &= \cos \phi \sin x + \sin \phi \cos x \\ &= \sin(x + \phi) = \sin\left(x + \frac{\pi}{3}\right) \end{aligned}$$

We are able to do this because the coefficients $\frac{1}{2}$ and $\sqrt{3}/2$ are precisely the cosine and sine of a particular number, in this case, $\pi/3$. We can use this same idea in general to write $A \sin x + B \cos x$ in the form $k \sin(x + \phi)$. We start by multiplying the numerator and denominator by $\sqrt{A^2 + B^2}$ to get

$$A \sin x + B \cos x = \sqrt{A^2 + B^2} \left(\frac{A}{\sqrt{A^2 + B^2}} \sin x + \frac{B}{\sqrt{A^2 + B^2}} \cos x \right)$$

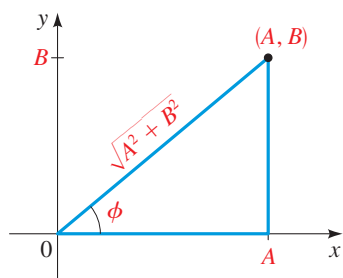


FIGURE 4

We need a number ϕ with the property that

$$\cos \phi = \frac{A}{\sqrt{A^2 + B^2}} \quad \text{and} \quad \sin \phi = \frac{B}{\sqrt{A^2 + B^2}}$$

Figure 4 shows that the point (A, B) in the plane determines a number ϕ with precisely this property. With this ϕ we have

$$\begin{aligned} A \sin x + B \cos x &= \sqrt{A^2 + B^2} (\cos \phi \sin x + \sin \phi \cos x) \\ &= \sqrt{A^2 + B^2} \sin(x + \phi) \end{aligned}$$

We have proved the following theorem.

SUMS OF SINES AND COSINES

If A and B are real numbers, then

$$A \sin x + B \cos x = k \sin(x + \phi)$$

where $k = \sqrt{A^2 + B^2}$ and ϕ satisfies

$$\cos \phi = \frac{A}{\sqrt{A^2 + B^2}} \quad \text{and} \quad \sin \phi = \frac{B}{\sqrt{A^2 + B^2}}$$

EXAMPLE 8 ■ A Sum of Sine and Cosine Terms

Express $3 \sin x + 4 \cos x$ in the form $k \sin(x + \phi)$.

SOLUTION By the preceding theorem, $k = \sqrt{A^2 + B^2} = \sqrt{3^2 + 4^2} = 5$. The angle ϕ has the property that $\sin \phi = B/k = \frac{4}{5}$ and $\cos \phi = A/k = \frac{3}{5}$, and ϕ in Quadrant I (because $\sin \phi$ and $\cos \phi$ are both positive), so $\phi = \sin^{-1} \frac{4}{5}$. Using a calculator, we get $\phi \approx 53.1^\circ$. Thus

$$3 \sin x + 4 \cos x \approx 5 \sin(x + 53.1^\circ)$$

Now Try Exercise 59

EXAMPLE 9 ■ Graphing a Trigonometric Function

Write the function $f(x) = -\sin 2x + \sqrt{3} \cos 2x$ in the form $k \sin(2x + \phi)$, and use the new form to graph the function.

SOLUTION Since $A = -1$ and $B = \sqrt{3}$, we have $k = \sqrt{A^2 + B^2} = \sqrt{1 + 3} = 2$. The angle ϕ satisfies $\cos \phi = -\frac{1}{2}$ and $\sin \phi = \frac{\sqrt{3}}{2}$. From the signs of these quantities we conclude that ϕ is in Quadrant II. Thus $\phi = 2\pi/3$. By the preceding theorem we can write

$$f(x) = -\sin 2x + \sqrt{3} \cos 2x = 2 \sin\left(2x + \frac{2\pi}{3}\right)$$

Using the form

$$f(x) = 2 \sin 2\left(x + \frac{\pi}{3}\right)$$

we see that the graph is a sine curve with amplitude 2, period $2\pi/2 = \pi$, and phase shift $-\pi/3$. The graph is shown in Figure 5.

Now Try Exercise 63

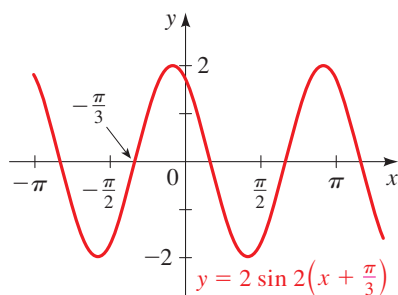


FIGURE 5

7.2 EXERCISES

CONCEPTS

- If we know the values of the sine and cosine of x and y , we can find the value of $\sin(x + y)$ by using the _____ Formula for Sine. State the formula:
 $\sin(x + y) = \underline{\hspace{2cm}}$.
- If we know the values of the sine and cosine of x and y , we can find the value of $\cos(x - y)$ by using the _____ Formula for Cosine. State the formula:
 $\cos(x - y) = \underline{\hspace{2cm}}$.

SKILLS

3–14 ■ Values of Trigonometric Functions Use an Addition or Subtraction Formula to find the exact value of the expression, as demonstrated in Example 1.

- $\sin 75^\circ$
- $\cos 105^\circ$
- $\tan 15^\circ$
- $\sin 15^\circ$
- $\cos 195^\circ$
- $\tan 165^\circ$
- $\sin \frac{19\pi}{12}$
- $\cos \frac{17\pi}{12}$
- $\tan\left(-\frac{\pi}{12}\right)$
- $\sin\left(-\frac{5\pi}{12}\right)$
- $\cos \frac{11\pi}{12}$
- $\tan \frac{7\pi}{12}$

15–20 ■ Values of Trigonometric Functions Use an Addition or Subtraction Formula to write the expression as a trigonometric function of one number, and then find its exact value.

- $\sin 18^\circ \cos 27^\circ + \cos 18^\circ \sin 27^\circ$
- $\cos 10^\circ \cos 80^\circ - \sin 10^\circ \sin 80^\circ$
- $\cos \frac{3\pi}{7} \cos \frac{2\pi}{21} + \sin \frac{3\pi}{7} \sin \frac{2\pi}{21}$
- $\frac{\tan \frac{\pi}{18} + \tan \frac{\pi}{9}}{1 - \tan \frac{\pi}{18} \tan \frac{\pi}{9}}$
- $\frac{\tan 73^\circ - \tan 13^\circ}{1 + \tan 73^\circ \tan 13^\circ}$
- $\cos \frac{13\pi}{15} \cos\left(-\frac{\pi}{5}\right) - \sin \frac{13\pi}{15} \sin\left(-\frac{\pi}{5}\right)$

21–24 ■ Cofunction Identities Prove the cofunction identity using the Addition and Subtraction Formulas.

- $\tan\left(\frac{\pi}{2} - u\right) = \cot u$
- $\cot\left(\frac{\pi}{2} - u\right) = \tan u$
- $\sec\left(\frac{\pi}{2} - u\right) = \csc u$
- $\csc\left(\frac{\pi}{2} - u\right) = \sec u$

25–46 ■ Proving Identities Prove the identity.

- $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$
- $\cos\left(x - \frac{\pi}{2}\right) = \sin x$
- $\sin(x - \pi) = -\sin x$
- $\cos(x - \pi) = -\cos x$
- $\tan(x - \pi) = \tan x$
- $\tan\left(x - \frac{\pi}{2}\right) = -\cot x$
- $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
- $\cos\left(x + \frac{\pi}{3}\right) + \sin\left(x - \frac{\pi}{6}\right) = 0$
- $\tan\left(x + \frac{\pi}{3}\right) = \frac{\sqrt{3} + \tan x}{1 - \sqrt{3} \tan x}$
- $\tan\left(x - \frac{\pi}{4}\right) = \frac{\tan x - 1}{\tan x + 1}$
- $\sin(x + y) - \sin(x - y) = 2 \cos x \sin y$
- $\cos(x + y) + \cos(x - y) = 2 \cos x \cos y$
- $\cot(x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$
- $\cot(x + y) = \frac{\cot x \cot y - 1}{\cot x + \cot y}$
- $\tan x - \tan y = \frac{\sin(x - y)}{\cos x \cos y}$
- $1 - \tan x \tan y = \frac{\cos(x + y)}{\cos x \cos y}$
- $\frac{\tan x - \tan y}{1 - \tan x \tan y} = \frac{\sin(x - y)}{\cos(x + y)}$
- $\frac{\sin(x + y) - \sin(x - y)}{\cos(x + y) + \cos(x - y)} = \tan y$
- $\cos(x + y) \cos(x - y) = \cos^2 x - \sin^2 y$
- $\cos(x + y) \cos y + \sin(x + y) \sin y = \cos x$
- $\sin(x + y + z) = \sin x \cos y \cos z + \cos x \sin y \cos z + \cos x \cos y \sin z - \sin x \sin y \sin z$
- $\tan(x - y) + \tan(y - z) + \tan(z - x) = \tan(x - y) \tan(y - z) \tan(z - x)$
- $\cos(\sin^{-1} x - \tan^{-1} y)$
- $\tan(\sin^{-1} x + \cos^{-1} y)$
- $\sin(\tan^{-1} x - \tan^{-1} y)$
- $\sin(\sin^{-1} x + \cos^{-1} y)$

51–54 ■ Expressions Involving Inverse Trigonometric Functions Find the exact value of the expression.

51. $\sin(\cos^{-1} \frac{1}{2} + \tan^{-1} 1)$ 52. $\cos(\sin^{-1} \frac{\sqrt{3}}{2} + \cot^{-1} \sqrt{3})$
 53. $\tan(\sin^{-1} \frac{3}{4} - \cos^{-1} \frac{1}{3})$ 54. $\sin(\cos^{-1} \frac{2}{3} - \tan^{-1} \frac{1}{2})$

55–58 ■ Evaluating Expressions Involving Trigonometric Functions Evaluate each expression under the given conditions.

55. $\cos(\theta - \phi)$; $\cos \theta = \frac{3}{5}$, θ in Quadrant IV, $\tan \phi = -\sqrt{3}$, ϕ in Quadrant II.
 56. $\sin(\theta - \phi)$; $\tan \theta = \frac{4}{3}$, θ in Quadrant III, $\sin \phi = -\sqrt{10}/10$, ϕ in Quadrant IV.
 57. $\sin(\theta + \phi)$; $\sin \theta = \frac{5}{13}$, θ in Quadrant I, $\cos \phi = -2\sqrt{5}/5$, ϕ in Quadrant II.
 58. $\tan(\theta + \phi)$; $\cos \theta = -\frac{1}{3}$, θ in Quadrant III, $\sin \phi = \frac{1}{4}$, ϕ in Quadrant II.

59–62 ■ Expressions in Terms of Sine Write the expression in terms of sine only.

59. $-\sqrt{3} \sin x + \cos x$ 60. $\sin x - \cos x$
 61. $5(\sin 2x - \cos 2x)$ 62. $3 \sin \pi x + 3\sqrt{3} \cos \pi x$

63–64 ■ Graphing a Trigonometric Function (a) Express the function in terms of sine only. (b) Graph the function.

63. $g(x) = \cos 2x + \sqrt{3} \sin 2x$ 64. $f(x) = \sin x + \cos x$

SKILLS Plus**65–66 ■ Difference Quotient** Let $f(x) = \cos x$ and $g(x) = \sin x$. Use Addition or Subtraction Formulas to show the following.

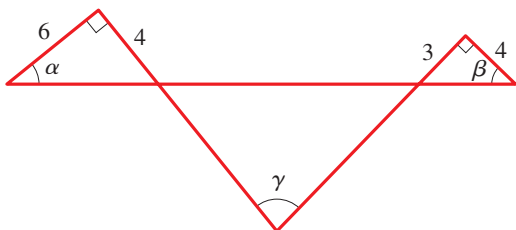
65. $\frac{f(x+h) - f(x)}{h} = -\cos x \left(\frac{1 - \cos h}{h} \right) - \sin x \left(\frac{\sin h}{h} \right)$
 66. $\frac{g(x+h) - g(x)}{h} = \left(\frac{\sin h}{h} \right) \cos x - \sin x \left(\frac{1 - \cos h}{h} \right)$

**67–68 ■ Discovering an Identity Graphically** In these exercises we discover an identity graphically and then prove the identity. (a) Graph the function and make a conjecture, then (b) prove that your conjecture is true.

67. $y = \sin^2\left(x + \frac{\pi}{4}\right) + \sin^2\left(x - \frac{\pi}{4}\right)$

68. $y = -\frac{1}{2}[\cos(x + \pi) + \cos(x - \pi)]$

69. Difference of Two Angles Show that if $\beta - \alpha = \pi/2$, then

$$\sin(x + \alpha) + \cos(x + \beta) = 0$$
70. Sum of Two Angles Refer to the figure. Show that $\alpha + \beta = \gamma$, and find $\tan \gamma$.**71–72 ■ Identities Involving Inverse Trigonometric Functions** Prove the identity.

71. $\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}x + \tan^{-1}y$

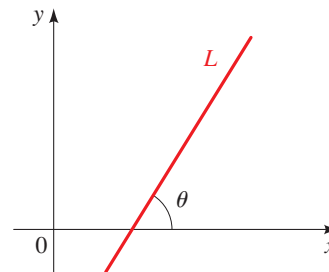
[Hint: Let $u = \tan^{-1}x$ and $v = \tan^{-1}y$, so that $x = \tan u$ and $y = \tan v$. Use an Addition Formula to find $\tan(u + v)$.]

72. $\tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$, $x > 0$ [Hint: Let $u = \tan^{-1}x$ and $v = \tan^{-1}\left(\frac{1}{x}\right)$, so that $x = \tan u$ and $\frac{1}{x} = \tan v$. Use an Addition Formula to find $\cot(u + v)$.]

73. Angle Between Two Lines In this exercise we find a formula for the angle formed by two lines in a coordinate plane.

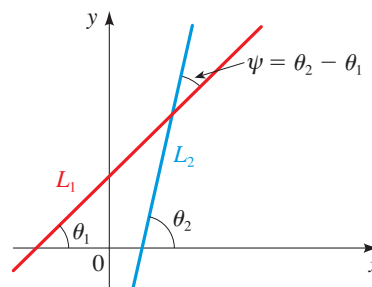
- (a) If L is a line in the plane and θ is the angle formed by the line and the x -axis as shown in the figure, show that the slope m of the line is given by

$$m = \tan \theta$$



- (b) Let L_1 and L_2 be two nonparallel lines in the plane with slopes m_1 and m_2 , respectively. Let ψ be the acute angle formed by the two lines (see the following figure). Show that

$$\tan \psi = \frac{m_2 - m_1}{1 + m_1 m_2}$$

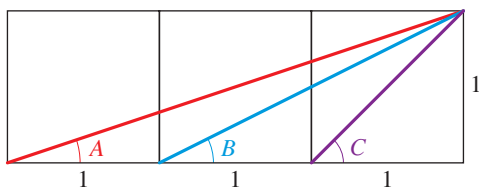


- (c) Find the acute angle formed by the two lines

$$y = \frac{1}{3}x + 1 \quad \text{and} \quad y = \frac{1}{2}x - 3$$

- (d) Show that if two lines are perpendicular, then the slope of one is the negative reciprocal of the slope of the other. [Hint: First find an expression for $\cot \psi$.]

74. Find $\angle A + \angle B + \angle C$ in the figure. [Hint: First use an Addition Formula to find $\tan(A + B)$.]



APPLICATIONS

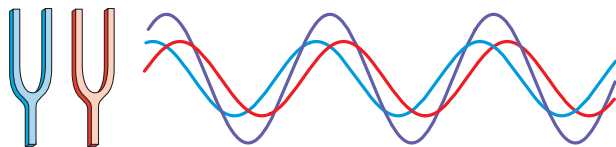


75. **Adding an Echo** A digital delay device echoes an input signal by repeating it a fixed length of time after it is received. If such a device receives the pure note $f_1(t) = 5 \sin t$ and echoes the pure note $f_2(t) = 5 \cos t$, then the combined sound is $f(t) = f_1(t) + f_2(t)$.

- (a) Graph $y = f(t)$, and observe that the graph has the form of a sine curve $y = k \sin(t + \phi)$.
 (b) Find k and ϕ .
76. **Interference** Two identical tuning forks are struck, one a fraction of a second after the other. The sounds produced are modeled by $f_1(t) = C \sin \omega t$ and $f_2(t) = C \sin(\omega t + \alpha)$. The two sound waves interfere to produce a single sound modeled by the sum of these functions

$$f(t) = C \sin \omega t + C \sin(\omega t + \alpha)$$

- (a) Use the Addition Formula for Sine to show that f can be written in the form $f(t) = A \sin \omega t + B \cos \omega t$, where A and B are constants that depend on α .
 (b) Suppose that $C = 10$ and $\alpha = \pi/3$. Find constants k and ϕ so that $f(t) = k \sin(\omega t + \phi)$.



DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

77. **PROVE: Addition Formula for Sine** In the text we proved only the Addition and Subtraction Formulas for Cosine. Use these formulas and the cofunction identities

$$\sin x = \cos\left(\frac{\pi}{2} - x\right)$$

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

to prove the Addition Formula for Sine. [Hint: To get started, use the first cofunction identity to write

$$\begin{aligned} \sin(s + t) &= \cos\left(\frac{\pi}{2} - (s + t)\right) \\ &= \cos\left(\left(\frac{\pi}{2} - s\right) - t\right) \end{aligned}$$

and use the Subtraction Formula for Cosine.]

78. **PROVE: Addition Formula for Tangent** Use the Addition Formulas for Cosine and Sine to prove the Addition Formula for Tangent. [Hint: Use

$$\tan(s + t) = \frac{\sin(s + t)}{\cos(s + t)}$$

and divide the numerator and denominator by $\cos s \cos t$.]

7.3 DOUBLE-ANGLE, HALF-ANGLE, AND PRODUCT-SUM FORMULAS

■ Double-Angle Formulas ■ Half-Angle Formulas ■ Evaluating Expressions Involving Inverse Trigonometric Functions ■ Product-Sum Formulas

The identities we consider in this section are consequences of the addition formulas. The **Double-Angle Formulas** allow us to find the values of the trigonometric functions at $2x$ from their values at x . The **Half-Angle Formulas** relate the values of the trigonometric functions at $\frac{1}{2}x$ to their values at x . The **Product-Sum Formulas** relate products of sines and cosines to sums of sines and cosines.

■ Double-Angle Formulas

The formulas in the box on the next page are immediate consequences of the addition formulas, which we proved in Section 7.2.