5.1 THE UNIT CIRCLE

■ The Unit Circle ■ Terminal Points on the Unit Circle ■ The Reference Number

In this section we explore some properties of the circle of radius 1 centered at the origin. These properties are used in the next section to define the trigonometric functions.

■ The Unit Circle

The set of points at a distance 1 from the origin is a circle of radius 1 (see Figure 1). In Section 1.9 we learned that the equation of this circle is $x^2 + y^2 = 1$.

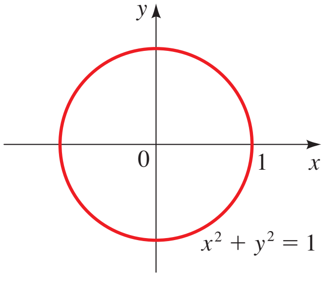


FIGURE 1 The unit circle

Circles are studied in Section 1.9, page 97.

THE UNIT CIRCLE

The **unit circle** is the circle of radius 1 centered at the origin in the xy-plane. Its equation is

$$x^2 + y^2 = 1$$

EXAMPLE 1 ■ A Point on the Unit Circle

Show that the point $P\left(\dfrac{\sqrt{3}}{3}, \dfrac{\sqrt{6}}{3}\right)$ is on the unit circle.

SOLUTION We need to show that this point satisfies the equation of the unit circle, that is, $x^2 + y^2 = 1$. Since

$$\left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{6}}{3}\right)^2 = \frac{3}{9} + \frac{6}{9} = 1$$

P is on the unit circle.

Now Try Exercise 3 ■

EXAMPLE 2 ■ Locating a Point on the Unit Circle

The point $P(\sqrt{3}/2, y)$ is on the unit circle in Quadrant IV. Find its y-coordinate.

SOLUTION Since the point is on the unit circle, we have

$$\left(\frac{\sqrt{3}}{2}\right)^2 + y^2 = 1$$

$$y^2 = 1 - \frac{3}{4} = \frac{1}{4}$$

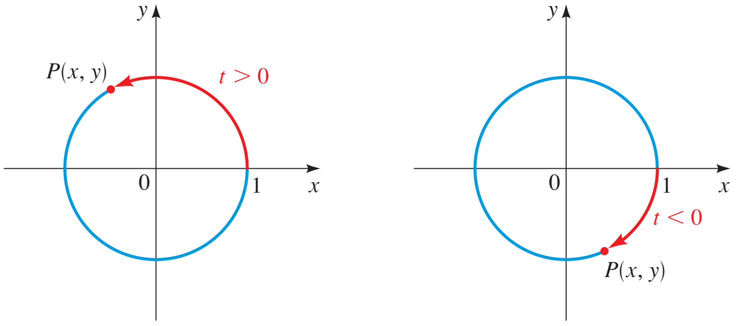
$$y = \pm\frac{1}{2}$$

Since the point is in Quadrant IV, its y-coordinate must be negative, so $y = -\frac{1}{2}$.

Now Try Exercise 9 ■

■ Terminal Points on the Unit Circle

Suppose t is a real number. If $t \geq 0$, let's mark off a distance t along the unit circle, starting at the point $(1, 0)$ and moving in a counterclockwise direction. If $t < 0$, we mark off a distance $|t|$ in a clockwise direction (Figure 2). In this way we arrive at a point $P(x, y)$ on the unit circle. The point $P(x, y)$ obtained in this way is called the **terminal point** determined by the real number t.



(a) Terminal point $P(x, y)$ determined by $t > 0$

(b) Terminal point $P(x, y)$ determined by $t < 0$

FIGURE 2

The circumference of the unit circle is $C = 2\pi(1) = 2\pi$. So if a point starts at $(1, 0)$ and moves counterclockwise all the way around the unit circle and returns to $(1, 0)$, it travels a distance of 2π. To move halfway around the circle, it travels a distance of $\frac{1}{2}(2\pi) = \pi$. To move a quarter of the distance around the circle, it travels a distance of $\frac{1}{4}(2\pi) = \pi/2$. Where does the point end up when it travels these distances along the circle? From Figure 3 we see, for example, that when it travels a distance of π starting at $(1, 0)$, its terminal point is $(-1, 0)$.

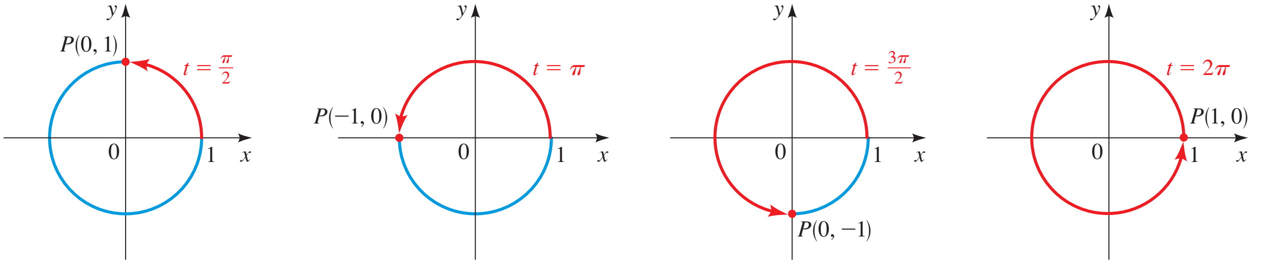


FIGURE 3 Terminal points determined by $t = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π

EXAMPLE 3 ■ Finding Terminal Points

Find the terminal point on the unit circle determined by each real number t.

(a) $t = 3\pi$ **(b)** $t = -\pi$ **(c)** $t = -\dfrac{\pi}{2}$

SOLUTION From Figure 4 we get the following:

(a) The terminal point determined by 3π is $(-1, 0)$.

(b) The terminal point determined by $-\pi$ is $(-1, 0)$.

(c) The terminal point determined by $-\pi/2$ is $(0, -1)$.

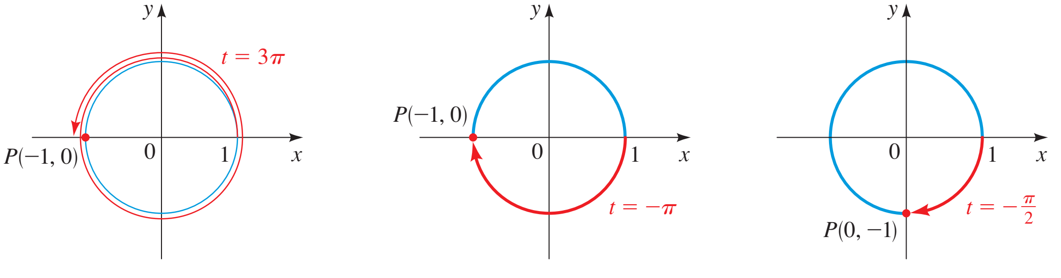


FIGURE 4

Notice that different values of t can determine the same terminal point.

Now Try Exercise 23 ■

The terminal point $P(x, y)$ determined by $t = \pi/4$ is the same distance from $(1, 0)$ as from $(0, 1)$ along the unit circle (see Figure 5).

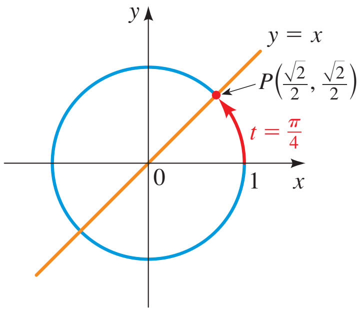


FIGURE 5

Since the unit circle is symmetric with respect to the line $y = x$, it follows that P lies on the line $y = x$. So P is the point of intersection (in the Quadrant I) of the circle $x^2 + y^2 = 1$ and the line $y = x$. Substituting x for y in the equation of the circle, we get

$$x^2 + x^2 = 1$$

$$2x^2 = 1 \qquad \text{Combine like terms}$$

$$x^2 = \frac{1}{2} \qquad \text{Divide by 2}$$

$$x = \pm\frac{1}{\sqrt{2}} \qquad \text{Take square roots}$$

Since P is in the Quadrant I, $x = 1/\sqrt{2}$ and since $y = x$, we have $y = 1/\sqrt{2}$ also. Thus the terminal point determined by $\pi/4$ is

$$P\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

Similar methods can be used to find the terminal points determined by $t = \pi/6$ and $t = \pi/3$ (see Exercises 61 and 62). Table 1 and Figure 6 give the terminal points for some special values of t.

TABLE 1

t	**Terminal point determined by t**
0	$(1, 0)$
$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{3}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2}$	$(0, 1)$

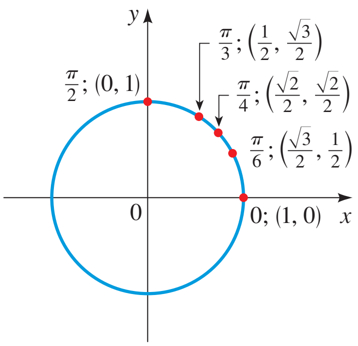


FIGURE 6

EXAMPLE 4 ■ Finding Terminal Points

Find the terminal point determined by each given real number t.

(a) $t = -\frac{\pi}{4}$ **(b)** $t = \frac{3\pi}{4}$ **(c)** $t = -\frac{5\pi}{6}$

SOLUTION

(a) Let P be the terminal point determined by $-\pi/4$, and let Q be the terminal point determined by $\pi/4$. From Figure 7(a) we see that the point P has the same coordinates as Q except for sign. Since P is in Quadrant IV, its x-coordinate is positive and its y-coordinate is negative. Thus, the terminal point is $P\left(\sqrt{2}/2, -\sqrt{2}/2\right)$.

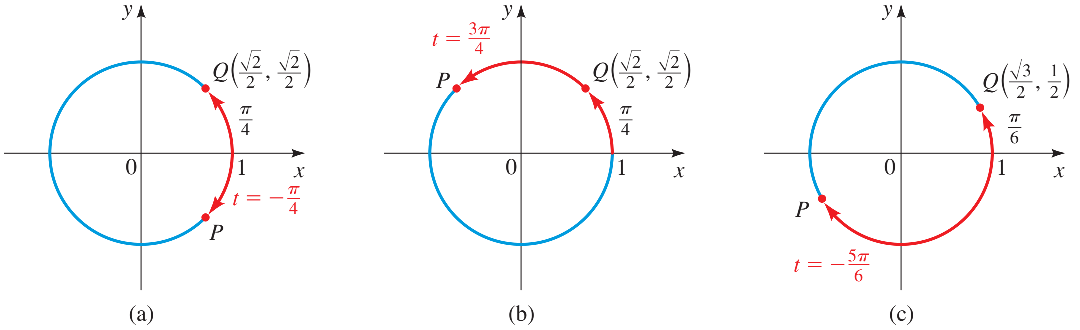


FIGURE 7

(b) Let P be the terminal point determined by $3\pi/4$, and let Q be the terminal point determined by $\pi/4$. From Figure 7(b) we see that the point P has the same coordinates as Q except for sign. Since P is in Quadrant II, its x-coordinate is negative and its y-coordinate is positive. Thus the terminal point is $P\left(-\sqrt{2}/2, \sqrt{2}/2\right)$.

(c) Let P be the terminal point determined by $-5\pi/6$, and let Q be the terminal point determined by $\pi/6$. From Figure 7(c) we see that the point P has the same coordinates as Q except for sign. Since P is in Quadrant III, its coordinates are both negative. Thus the terminal point is $P\left(-\sqrt{3}/2, -\frac{1}{2}\right)$.

Now Try Exercise 27 ■

The Reference Number

From Examples 3 and 4 we see that to find a terminal point in any quadrant we need only know the "corresponding" terminal point in the first quadrant. We use the idea of the *reference number* to help us find terminal points.

REFERENCE NUMBER

Let t be a real number. The **reference number** $\bar{t}$ associated with t is the shortest distance along the unit circle between the terminal point determined by t and the x-axis.

Figure 8 shows that to find the reference number $\bar{t}$, it's helpful to know the quadrant in which the terminal point determined by t lies. If the terminal point lies in Quadrant I or IV, where x is positive, we find $\bar{t}$ by moving along the circle to the *positive* x-axis. If it lies in Quadrant II or III, where x is negative, we find $\bar{t}$ by moving along the circle to the *negative* x-axis.

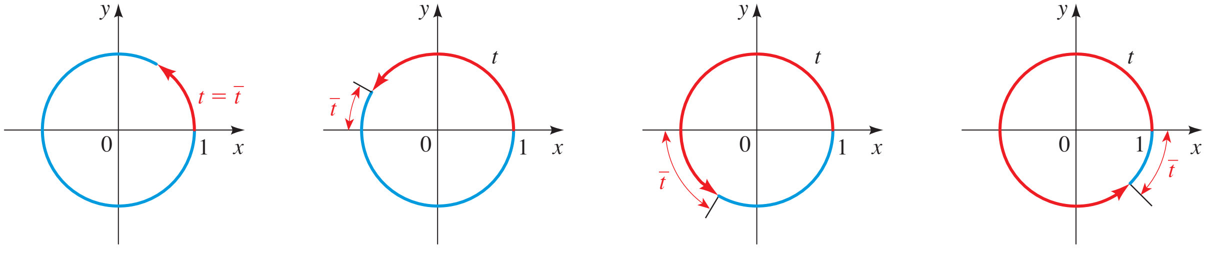


FIGURE 8 The reference number $\bar{t}$ for t

EXAMPLE 5 ■ Finding Reference Numbers

Find the reference number for each value of t.

(a) $t = \dfrac{5\pi}{6}$ **(b)** $t = \dfrac{7\pi}{4}$ **(c)** $t = -\dfrac{2\pi}{3}$ **(d)** $t = 5.80$

SOLUTION From Figure 9 we find the reference numbers as follows.

(a) $\bar{t} = \pi - \dfrac{5\pi}{6} = \dfrac{\pi}{6}$ **(b)** $\bar{t} = 2\pi - \dfrac{7\pi}{4} = \dfrac{\pi}{4}$

(c) $\bar{t} = \pi - \dfrac{2\pi}{3} = \dfrac{\pi}{3}$ **(d)** $\bar{t} = 2\pi - 5.80 \approx 0.48$

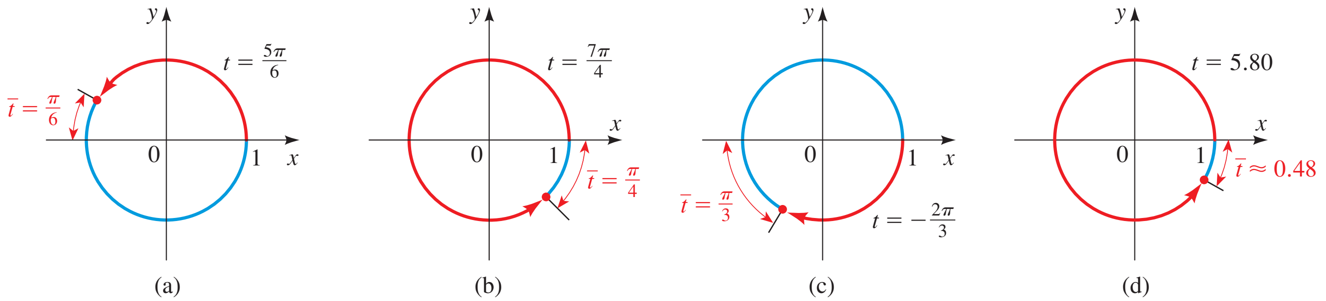


FIGURE 9

Now Try Exercise 37 ■

USING REFERENCE NUMBERS TO FIND TERMINAL POINTS

To find the terminal point P determined by any value of t, we use the following steps:

1. Find the reference number $\bar{t}$.
2. Find the terminal point $Q(a, b)$ determined by $\bar{t}$.
3. The terminal point determined by t is $P(\pm a, \pm b)$, where the signs are chosen according to the quadrant in which this terminal point lies.

EXAMPLE 6 ■ Using Reference Numbers to Find Terminal Points

Find the terminal point determined by each given real number t.

(a) $t = \dfrac{5\pi}{6}$ **(b)** $t = \dfrac{7\pi}{4}$ **(c)** $t = -\dfrac{2\pi}{3}$

SOLUTION The reference numbers associated with these values of t were found in Example 5.

(a) The reference number is $\bar{t} = \pi/6$, which determines the terminal point $\left(\sqrt{3}/2, \frac{1}{2}\right)$ from Table 1. Since the terminal point determined by t is in Quadrant II, its x-coordinate is negative and its y-coordinate is positive. Thus the desired terminal point is

$$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

(b) The reference number is $\bar{t} = \pi/4$, which determines the terminal point $\left(\sqrt{2}/2, \sqrt{2}/2\right)$ from Table 1. Since the terminal point is in Quadrant IV, its x-coordinate is positive and its y-coordinate is negative. Thus the desired terminal point is

$$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

(c) The reference number is $\bar{t} = \pi/3$, which determines the terminal point $\left(\frac{1}{2}, \sqrt{3}/2\right)$ from Table 1. Since the terminal point determined by t is in Quadrant III, its coordinates are both negative. Thus the desired terminal point is

$$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

Now Try Exercise 41 ■

Since the circumference of the unit circle is 2π, the terminal point determined by t is the same as that determined by $t + 2\pi$ or $t - 2\pi$. In general, we can add or subtract 2π any number of times without changing the terminal point determined by t. We use this observation in the next example to find terminal points for large t.

EXAMPLE 7 ■ Finding the Terminal Point for Large t

Find the terminal point determined by $t = \dfrac{29\pi}{6}$.

SOLUTION Since

$$t = \frac{29\pi}{6} = 4\pi + \frac{5\pi}{6}$$

we see that the terminal point of t is the same as that of $5\pi/6$ (that is, we subtract 4π). So by Example 6(a) the terminal point is $\left(-\sqrt{3}/2, \frac{1}{2}\right)$. (See Figure 10.)

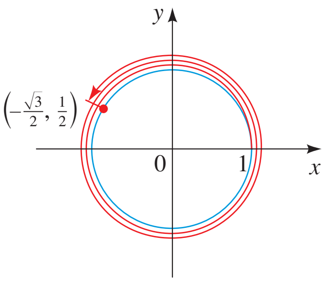


FIGURE 10

Now Try Exercise 47 ■

5.1 EXERCISES

CONCEPTS

1. (a) The unit circle is the circle centered at ________ with radius ________.

(b) The equation of the unit circle is ____________.

(c) Suppose the point $P(x, y)$ is on the unit circle. Find the missing coordinate:

(i) $P(1, \square)$ (ii) $P(\square, 1)$

(iii) $P(-1, \square)$ (iv) $P(\square, -1)$

2. (a) If we mark off a distance t along the unit circle, starting at $(1, 0)$ and moving in a counterclockwise direction, we arrive at the ____________ point determined by t.

(b) The terminal points determined by $\pi/2$, π, $-\pi/2$, 2π are ________, ________, ________, and ________, respectively.

SKILLS

3–8 ■ Points on the Unit Circle Show that the point is on the unit circle.

3. $\left(\frac{3}{5}, -\frac{4}{5}\right)$ **4.** $\left(-\frac{24}{25}, -\frac{7}{25}\right)$

5. $\left(\frac{3}{4}, -\frac{\sqrt{7}}{4}\right)$ **6.** $\left(-\frac{5}{7}, -\frac{2\sqrt{6}}{7}\right)$

7. $\left(-\frac{\sqrt{5}}{3}, \frac{2}{3}\right)$ **8.** $\left(\frac{\sqrt{11}}{6}, \frac{5}{6}\right)$

9–14 ■ Points on the Unit Circle Find the missing coordinate of P, using the fact that P lies on the unit circle in the given quadrant.

Coordinates	Quadrant
9. $P\left(-\frac{3}{5}, \square\right)$	III
10. $P\left(\square, -\frac{7}{25}\right)$	IV
11. $P\left(\square, \frac{1}{3}\right)$	II

Coordinates	Quadrant
12. $P\left(\frac{2}{5}, \square\right)$	I
13. $P\left(\square, -\frac{2}{7}\right)$	IV
14. $P\left(-\frac{2}{3}, \square\right)$	II

15–20 ■ Points on the Unit Circle The point P is on the unit circle. Find $P(x, y)$ from the given information.

15. The x-coordinate of P is $\frac{5}{13}$, and the y-coordinate is negative.

16. The y-coordinate of P is $-\frac{3}{5}$, and the x-coordinate is positive.

17. The y-coordinate of P is $\frac{2}{3}$, and the x-coordinate is negative.

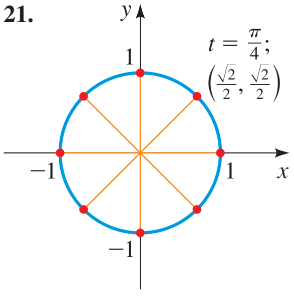
18. The x-coordinate of P is positive, and the y-coordinate of P is $-\sqrt{5}/5$.

19. The x-coordinate of P is $-\sqrt{2}/3$, and P lies below the x-axis.

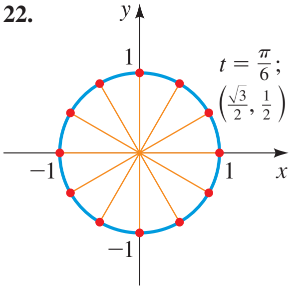
20. The x-coordinate of P is $-\frac{2}{5}$, and P lies above the x-axis.

21–22 ■ Terminal Points Find t and the terminal point determined by t for each point in the figure. In Exercise 21, t increases in increments of $\pi/4$; in Exercise 22, t increases in increments of $\pi/6$.

21.



22.



23–36 ■ Terminal Points Find the terminal point $P(x, y)$ on the unit circle determined by the given value of t.

23. $t = 4\pi$ **24.** $t = -3\pi$

25. $t = \dfrac{3\pi}{2}$ **26.** $t = \dfrac{5\pi}{2}$

27. $t = -\dfrac{\pi}{6}$ **28.** $t = \dfrac{7\pi}{6}$

29. $t = \dfrac{5\pi}{4}$ **30.** $t = \dfrac{4\pi}{3}$

31. $t = -\dfrac{7\pi}{6}$ **32.** $t = \dfrac{5\pi}{3}$

33. $t = -\dfrac{7\pi}{4}$ **34.** $t = -\dfrac{4\pi}{3}$

35. $t = -\dfrac{3\pi}{4}$ **36.** $t = \dfrac{11\pi}{6}$

37–40 ■ Reference Numbers Find the reference number for each value of t.

37. **(a)** $t = \dfrac{4\pi}{3}$ **(b)** $t = \dfrac{5\pi}{3}$

(c) $t = -\dfrac{7\pi}{6}$ **(d)** $t = 3.5$

38. **(a)** $t = 9\pi$ **(b)** $t = -\dfrac{5\pi}{4}$

(c) $t = \dfrac{25\pi}{6}$ **(d)** $t = 4$

39. **(a)** $t = \dfrac{5\pi}{7}$ **(b)** $t = -\dfrac{7\pi}{9}$

(c) $t = -3$ **(d)** $t = 5$

40. **(a)** $t = \dfrac{11\pi}{5}$ **(b)** $t = -\dfrac{9\pi}{7}$

(c) $t = 6$ **(d)** $t = -7$

41–54 ■ Terminal Points and Reference Numbers Find **(a)** the reference number for each value of t and **(b)** the terminal point determined by t.

41. $t = \dfrac{11\pi}{6}$ **42.** $t = \dfrac{2\pi}{3}$

43. $t = -\dfrac{4\pi}{3}$ **44.** $t = \dfrac{5\pi}{3}$

45. $t = -\dfrac{2\pi}{3}$ **46.** $t = -\dfrac{7\pi}{6}$

47. $t = \dfrac{13\pi}{4}$ **48.** $t = \dfrac{13\pi}{6}$

49. $t = \dfrac{41\pi}{6}$ **50.** $t = \dfrac{17\pi}{4}$

51. $t = -\dfrac{11\pi}{3}$ **52.** $t = \dfrac{31\pi}{6}$

53. $t = \dfrac{16\pi}{3}$ **54.** $t = -\dfrac{41\pi}{4}$

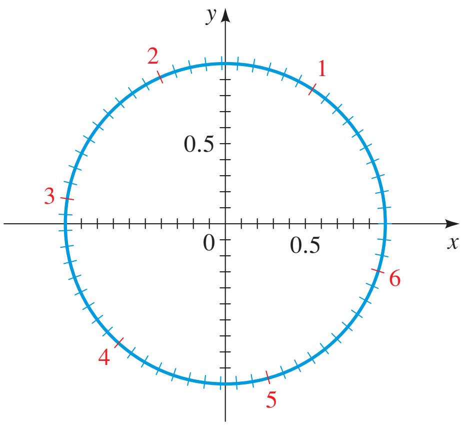
55–58 ■ Terminal Points The unit circle is graphed in the figure below. Use the figure to find the terminal point determined by the real number t, with coordinates rounded to one decimal place.

55. $t = 1$

56. $t = 2.5$

57. $t = -1.1$

58. $t = 4.2$



SKILLS Plus

59. Terminal Points Suppose that the terminal point determined by t is the point $\left(\frac{3}{5}, \frac{4}{5}\right)$ on the unit circle. Find the terminal point determined by each of the following.

(a) $\pi - t$ **(b)** $-t$

(c) $\pi + t$ **(d)** $2\pi + t$

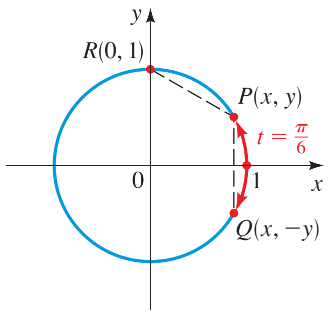
60. Terminal Points Suppose that the terminal point determined by t is the point $\left(\frac{3}{4}, \sqrt{7}/4\right)$ on the unit circle. Find the terminal point determined by each of the following.

(a) $-t$ **(b)** $4\pi + t$

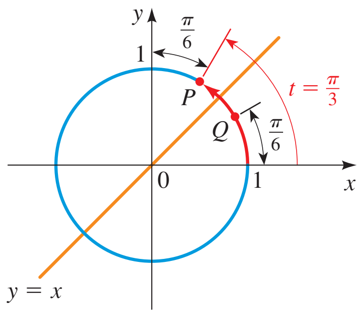
(c) $\pi - t$ **(d)** $t - \pi$

DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

61. DISCOVER ■ PROVE: Finding the Terminal Point for $\pi/6$ Suppose the terminal point determined by $t = \pi/6$ is $P(x, y)$ and the points Q and R are as shown in the figure. Why are the distances PQ and PR the same? Use this fact, together with the Distance Formula, to show that the coordinates of P satisfy the equation $2y = \sqrt{x^2 + (y - 1)^2}$. Simplify this equation using the fact that $x^2 + y^2 = 1$. Solve the simplified equation to find $P(x, y)$.



62. DISCOVER ■ PROVE: Finding the Terminal Point for $\pi/3$ Now that you know the terminal point determined by $t = \pi/6$, use symmetry to find the terminal point determined by $t = \pi/3$ (see the figure). Explain your reasoning.



5.2 TRIGONOMETRIC FUNCTIONS OF REAL NUMBERS

■ The Trigonometric Functions ■ Values of the Trigonometric Functions
■ Fundamental Identities

A function is a rule that assigns to each real number another real number. In this section we use properties of the unit circle from the preceding section to define the trigonometric functions.

■ The Trigonometric Functions

Recall that to find the terminal point $P(x, y)$ for a given real number t, we move a distance $|t|$ along the unit circle, starting at the point $(1, 0)$. We move in a counterclockwise direction if t is positive and in a clockwise direction if t is negative (see Figure 1). We now use the x- and y-coordinates of the point $P(x, y)$ to define several functions. For instance, we define the function called *sine* by assigning to each real number t the y-coordinate of the terminal point $P(x, y)$ determined by t. The functions *cosine*, *tangent*, *cosecant*, *secant*, and *cotangent* are also defined by using the coordinates of $P(x, y)$.

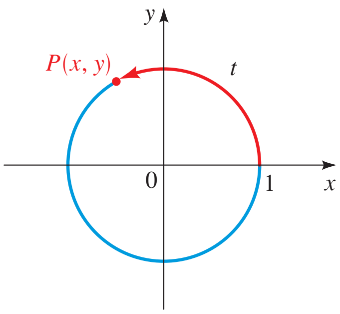


FIGURE 1

DEFINITION OF THE TRIGONOMETRIC FUNCTIONS

Let t be any real number and let $P(x, y)$ be the terminal point on the unit circle determined by t. We define

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x} \quad (x \neq 0)$$

$$\csc t = \frac{1}{y} \quad (y \neq 0) \qquad \sec t = \frac{1}{x} \quad (x \neq 0) \qquad \cot t = \frac{x}{y} \quad (y \neq 0)$$

Because the trigonometric functions can be defined in terms of the unit circle, they are sometimes called the **circular functions**.