Exam 1 Practice Questions

1. For $n = 0, 1, 2, \dots$ let

$$S_n = \sum_{k=0}^{n} \frac{2^{2k+1}}{5^k}.$$

(a) Give a linear iterative equation for S_{n+1} in terms of S_n, and state the initial value S_0.

(b) Is the equation from part (a) autonomous? Homogeneous?

(c) What is $\lim_{n\to\infty} S_n$?

(a) $$S_{n+1} = \sum_{k=0}^{n+1} \frac{2^{2k+1}}{5^k} = \sum_{k=0}^{n} \frac{2^{2k+1}}{5^k} + \frac{2^{2(n+1)+1}}{5^{n+1}}$$

$$\boxed{S_{n+1} = S_n + \frac{2^{2n+3}}{5^{n+1}}}$$

(b) Autonomous? No

($x_{n+1} = a_n x_n + b_n$ is autonomous if there exist numbers a & b such that $a_n = a$ & $b_n = b$ for all n.)

Homogeneous? No

($x_{n+1} = a_n x_n + b_n$ is homogeneous if $b_n = 0$ for all n.)

(c) $$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{2^{2k+1}}{5^k} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{2 \cdot 4^k}{5^k} = 2 \cdot \underbrace{\lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{4}{5}\right)^k}_{\text{Geometric Series}}$$

$$= 2 \cdot \lim_{n\to\infty} \frac{1 - \left(\frac{4}{5}\right)^{n+1}}{1 - \frac{4}{5}} = \frac{2}{1 - \frac{4}{5}} = 10$$

2. For $n = 1, 2, 3, \ldots$ let
$$P_n = \prod_{k=1}^{n} \left(1 + \frac{1}{k}\right).$$
   (a) Give a linear iterative equation for P_{n+1} in terms of P_n, and state the initial value P_1.
   (b) Is the equation from part (a) autonomous? Homogeneous?
   (c) Give a *simplified* exact solution for P_n.

(a) $P_{n+1} = \prod_{k=1}^{n+1}\left(1+\frac{1}{k}\right) = \left(1+\frac{1}{n+1}\right)\prod_{k=1}^{n}\left(1-\frac{1}{k}\right)$

$$\boxed{P_{n+1} = \left(1+\frac{1}{n+1}\right)P_n}$$

(b) Autonomous? No

Homogeneous? Yes

(c) $P_n = \prod_{k=1}^{n}\left(1+\frac{1}{k}\right) = \prod_{k=1}^{n}\frac{k+1}{k}$

$P_n = \frac{\cancel{2}}{1}\cdot\frac{\cancel{3}}{\cancel{2}}\cdot\frac{\cancel{4}}{\cancel{3}}\cdot\frac{\cancel{5}}{\cancel{4}}\cdot \cancel{\cdots}\cdot\frac{n+1}{\cancel{n}}$

$P_n = n+1$

3. (a) Rewrite the following sum using Σ-notation.

$$(0 \cdot 1 \cdot 2) + (1 \cdot 2 \cdot 3) + (2 \cdot 3 \cdot 4) + (3 \cdot 4 \cdot 5) + \cdots + (99 \cdot 100 \cdot 101)$$

(b) Use the following formulas to evaluate the sum from part (b).

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

(a) $\boxed{\sum_{k=1}^{100} (k-1)k(k+1)}$

(b) $$\sum_{k=1}^{100} (k-1)k(k+1) = \sum_{k=1}^{100} (k^3 - k) = \sum_{k=1}^{100} k^3 - \sum_{k=1}^{100} k$$

$$= \left(\frac{100(100+1)}{2}\right)^2 - \frac{100(100+1)}{2} = 5050^2 - 5050$$

$$= 25,497,450$$

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4. A woman purchases a \$28,900 car by paying \$5,000 in cash and borrowing a loan for the remianing balance. The loan charges an annual interest rate of 8.4% and is compounded monthly. Suppose the woman makes regular monthly payments (at the end of each compound period) of R dollars. Let D_n represent the woman's debt n months after borrowing the loan.
   (a) Construct an iterative equation that models how the woman's debt changes from one month to the next, and state the initial value.
   (b) If the woman wants to pay off her loan in exactly 6 years, what should her monthly payment R be? Round your answer to the nearest penny.

(a)

$$D_{n+1} = \left(1 + \frac{.084}{12}\right) D_n - R$$

$$D_0 = 28{,}900 - 5{,}000$$

$$\Rightarrow \boxed{\begin{aligned} D_{n+1} &= 1.007\, D_n - R \\ D_0 &= 23{,}900 \end{aligned}}$$

(b) EXACT SOLUTION:

$$D_n = 1.007^n D_0 - \left(\frac{1.007^n - 1}{1.007 - 1}\right) R$$

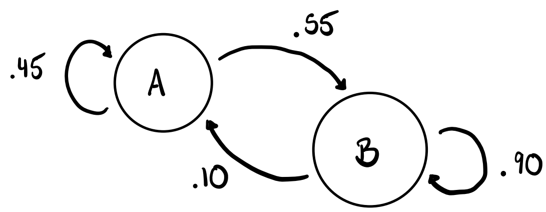
6 YEARS = 72 MONTHS

$$D_{72} = 0 \Rightarrow 1.007^{72}(23{,}900) - \left(\frac{1.007^{72} - 1}{1.007 - 1}\right) R = 0$$

$$R = \frac{1.007^{72}(23{,}900)}{\left(\frac{1.007^{72} - 1}{1.007 - 1}\right)} = \$423.73$$

5. Recently, a company began offering free virtual yoga classes to its employees on Wednesday mornings. Through surveys, the company discovered that 45% of employees that attend one week will attend the next week, and 10% of employees that do not attend one week will attend the next week.
   (a) Draw a transition diagram that summarizes the survey data.
   (b) Construct an iterative equation that models how the proportion of employees that attend Wednesday morning yoga class changes from one week to the next.
   (c) If 84% of employees attend class initially, what percentage of employees will attend class in 4 weeks? Round your answer to the nearest hundredth of a percent.
   (d) In the long run, what percentage of employees should the company expect to attend Wednesday morning yoga classes? Round your answer to the nearest hundredth of a percent.

(a)



(b)

$$A_{n+1} = .45A_n + .10B_n = .45A_n + .10(1 - A_n)$$

$$\boxed{A_{n+1} = .35A_n + .10}$$

(c)

$$A_0 = .84$$
$$A_1 = .35(.84) + .10 = .394$$
$$A_2 = .35(.394) + .10 = .2379$$
$$A_3 = .35(.2379) + .10 = .183265$$
$$A_4 = .35(.183265) + .10 = .16414275 \rightsquigarrow 16.41\%$$

$$\left[A_4 = .35^4(.84) + \left(\frac{1 - .35^4}{1 - .35}\right)(.10) \right]$$

(d) Exact Solution: $A_n = .35^n A_0 + \left(\frac{1 - .35^n}{1 - .35}\right)(.10)$

$$\lim_{n\to\infty} A_n = \frac{.10}{1 - .35} \approx .1538, \quad 15.38\%$$

6. It is known that the population of wild bison in Montana can be modeled by a linear equation, and its growth ratio (in one year) is 1.013. At the end of each year, 200 bison are hunted legally.
   (a) Find an iterative equation that models how the population of bison changes from one year to the next.
   (b) Find the fixed point of the population. Round your answer to the nearest integer.
   (c) Is the fixed point in (b) stable or unstable? Briefly justify your answer.
   (d) Now suppose that the government of Montana wants to change the number of legally hunted bison in order to fix the population at the current level of 30,000. How many legal huntings should be allowed? Round your answer to the nearest integer.

(a) $\boxed{P_{n+1} = 1.013\, P_n - 200}$

(b) $p = 1.013\, p - 200 \Rightarrow 200 = .013\, p$

$$p = \frac{200}{.013} \approx 15{,}385$$

(c) UNSTABLE BECAUSE THE COEFFICIENT 1.013 HAS ABSOLUTE VALUE GREATER THAN 1.

(d) $P_{n+1} = 1.013\, P_n - b$

FIND b SUCH THAT IF $P_0 = 30{,}000$ THEN $P_1 = 30{,}000$.

$$30{,}000 = 1.013\,(30{,}000) - b$$

$$b = (1.013 - 1)(30{,}000)$$

$$b = 390$$

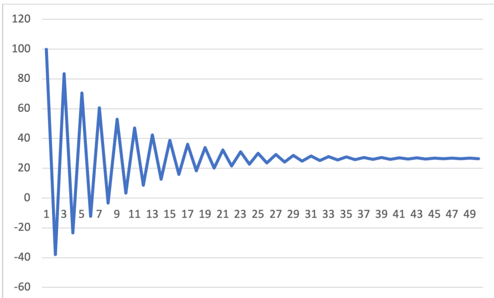
7. Match each of the following linear models with the line-grah of its solution.

(a) $x_{n+1} = 1.2x_n + 50$, $x_0 = 100$ 2

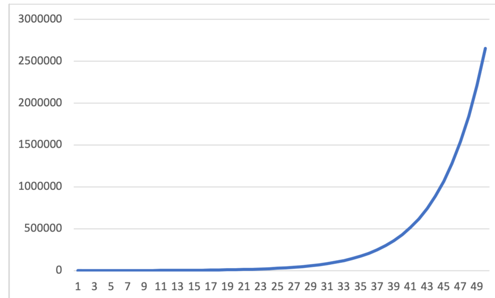
(b) $x_{n+1} = .88x_n + 50$, $x_0 = 100$ 4

(c) $x_{n+1} = -.88x_n + 50$, $x_0 = 100$ 1

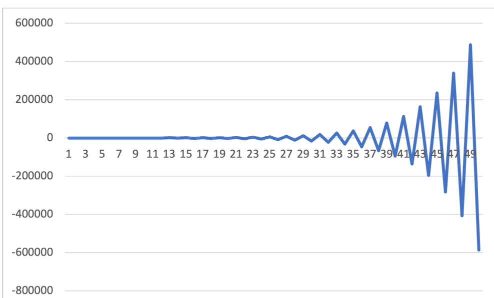
(d) $x_{n+1} = -1.2x_n + 50$, $x_0 = 100$ 3



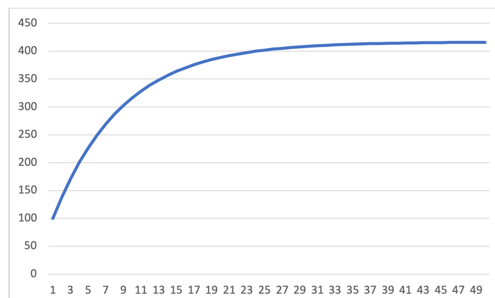
c Figure 1: (above)



a Figure 2: (above)



d Figure 3: (above)



b Figure 4: (above)

8. Use the Normal Equations

$$\left(\sum x_i^2\right) m + \left(\sum x_i\right) b = \sum x_i y_i$$
$$\left(\sum x_i\right) m + Nb = \sum y_i$$

to find the linear iterative model that best approximates the following observed values.

n	0	1	2	3	4	5
x_n	50	20	12	8	5	4

	i	x_i	$y_i = x_{i+1}$	x_i^2	$x_i y_i$
	0	50	20	2500	1000
	1	20	12	400	240
	2	12	8	144	96
	3	8	5	64	40
	4	5	4	25	20
TOTAL:		95	49	3133	1396

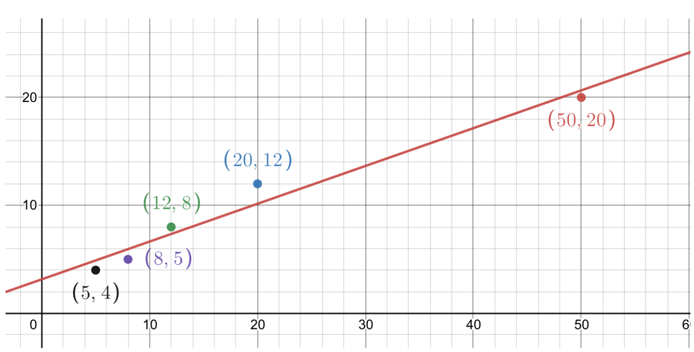
NORMAL EQUATIONS:

$$3133\,m + 95\,b = 1396$$
$$-19\left[\,95\,m + 5\,b = 49\,\right]$$
$$1328\,m = 465$$
$$m = \frac{465}{1328}$$

$$\Rightarrow\ 95\left(\frac{465}{1328}\right) + 5b = 49$$

$$b = \frac{49 - 95\left(\frac{465}{1328}\right)}{5} = \frac{20{,}897}{6640}$$

$$\therefore\ \boxed{x_{n+1} = \frac{465}{1328}\,x_n + \frac{20{,}897}{6640}} \quad \text{or} \quad x_{n+1} = .3502\,x_n + 3.1471$$



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