Homework 7

Solutions

1. (11 points) Let $z_1 = 3 + 7i$ and $z_2 = -2 + i$. Compute $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, $\frac{z_1}{z_2}$, $\overline{z_1}$, and $|z_1|$.

$$z_1 + z_2 = (3+7i) + (-2+i) = (3-2) + i(7+1) = \boxed{1+8i}$$

$$z_1 - z_2 = (3+7i) - (-2+i) = (3+2) + i(7-1) = \boxed{5+6i}$$

$$z_1 z_2 = (3+7i)(-2+i) = -6 + 3i - 14i + 7i^2 = (-6-7) + i(3-14) = \boxed{-13-11i}$$

$$\frac{z_1}{z_2} = \frac{3+7i}{-2+i} \cdot \frac{-2-i}{-2-i} = \frac{-6-3i-14i-7i^2}{4-i^2} = \frac{1-17i}{5} \text{ or } \boxed{\frac{1}{5} - \frac{17}{5}i}$$

$$\overline{z_1} = \overline{(3+7i)} = \boxed{3-7i}$$

$$|z_1| = \sqrt{3^2+7^2} = \boxed{\sqrt{58}}$$

2. Show that for any complex number $z = a + ib$, the following identities hold.

   (a) (4 points) $\frac{z+\overline{z}}{2} = \text{Re}(z)$ (i.e. the real part of z)

$$\frac{z+\overline{z}}{2} = \frac{a+ib+a-ib}{2} = \frac{2a}{2} = a = \text{Re}(a+ib) = \text{Re}(z) \checkmark$$

   (b) (4 points) $\frac{z-\overline{z}}{2i} = \text{Im}(z)$, (i.e. the imaginary part of z)

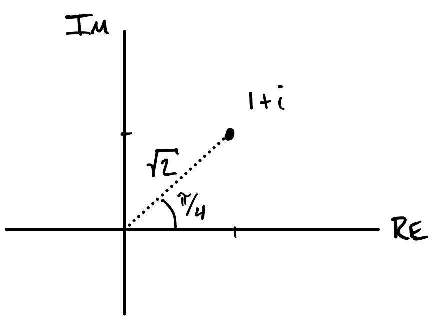
$$\frac{z-\overline{z}}{2i} = \frac{a+ib-(a-ib)}{2i} = \frac{i2b}{2i} = b = \text{Im}(a+ib) = \text{Im}(z) \checkmark$$

   (c) (4 points) $\sqrt{z\overline{z}} = |z|$

$$\sqrt{z\overline{z}} = \sqrt{(a+ib)(a-ib)} = \sqrt{a^2+b^2} = |a+ib| = |z| \checkmark$$

3. Write each expression in the form $a + ib$.
Hint: first rewrite the complex number in the polar form $r(\cos\theta + i\sin\theta)$, and then use De Moivre's Formula.

(a) (8 points) $(1+i)^7$



EULER'S FORMULA: $e^{i\theta} = \cos\theta + i\sin\theta$

$$1 + i = \sqrt{2}\cos\frac{\pi}{4} + i\sqrt{2}\sin\frac{\pi}{4}$$

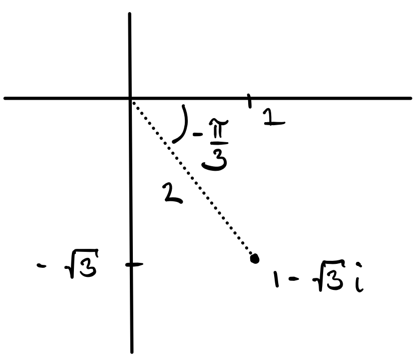
$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\,e^{i\frac{\pi}{4}}$$

$$\therefore\ (1+i)^7 = \left(\sqrt{2}\,e^{i\frac{\pi}{4}}\right)^7 = \sqrt{2}^{\,7} e^{i\frac{7\pi}{4}}$$

$$= 8\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = 8\sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$= \boxed{8 - 8i}$$

(b) (8 points) $(1-\sqrt{3}i)^5$



$$1 - i\sqrt{3} = 2\cos\left(-\frac{\pi}{3}\right) + i\,2\sin\left(-\frac{\pi}{3}\right)$$

$$= 2e^{i\left(-\frac{\pi}{3}\right)}$$

$$(1-i\sqrt{3})^5 = \left(2e^{i\left(-\frac{\pi}{3}\right)}\right)^5 = 2^5 e^{i\left(-\frac{5\pi}{3}\right)}$$

$$= 32\left(\cos\left(-\frac{5\pi}{3}\right) + i\sin\left(-\frac{5\pi}{3}\right)\right) = 32\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= \boxed{16 + i16\sqrt{3}}$$

4. Determine whether the origin is a sink, source, or saddle for the linear system

$$\vec{x}_{n+1} = A\vec{x}_n.$$

(a) (8 points) $A = \begin{bmatrix} 4/3 & -5 \\ 1/3 & -1 \end{bmatrix}$

FIND THE EIGENVALUES OF A:

CHARACTERISTIC EQ: $\left(\frac{4}{3} - r\right)(-1-r) + \frac{5}{3} = 0$

$-\frac{4}{3} - \frac{4}{3}r + r + r^2 + \frac{5}{3} = r^2 - \frac{1}{3}r + \frac{1}{3} = 0$

$3r^2 - r + 1 = 0 \implies r = \dfrac{1 \pm \sqrt{1-12}}{6} = \dfrac{1}{6} \pm i\dfrac{\sqrt{11}}{6}$

NOTE: $\sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{\sqrt{11}}{6}\right)^2} = \sqrt{\frac{12}{36}} < 1$

THAT IS, THE COMPLEX CONJUGATE EIGENVALUES HAVE MODULUS LESS THAN 1.
$\therefore$ THE ORIGIN IS A SINK.

(b) (8 points) $A = \begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix}$

$(5-r)(1-r) + 20 = 0$
$5 - 5r - r + r^2 + 20 = r^2 - 6r + 25 = 0$

$(r-3)^2 + 16 = 0 \implies r = 3 \pm \sqrt{-16} = 3 \pm 4i$

NOTE: $\sqrt{3^2 + 4^2} = 5 > 1$.

SINCE THE COMPLEX CONJUGATE EIGENVALUES HAVE MODULUS GREATER THAN 1, THE ORIGIN IS A SOURCE.

5. (15 points) Find the exact solution to the linear system

$$P_{n+1} = P_n - Q_n$$
$$Q_{n+1} = P_n + Q_n$$

when $P_0 = 8000$ and $Q_0 = 2000$.

SYSTEM: $\begin{bmatrix} P_{n+1} \\ Q_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_n \\ Q_n \end{bmatrix}$

FIRST FIND EIGENVALUES: $\det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = 0$

$$(1-\lambda)^2 + 1 = 0 \quad \Rightarrow \quad (1-\lambda)^2 = -1 \quad \Rightarrow \quad 1-\lambda = \pm i$$

$$\therefore \lambda = 1 \pm i = \sqrt{2}\left(\cos\frac{\pi}{4} \pm i \sin\frac{\pi}{4}\right)$$

$\therefore$ GENERAL SOLUTION HAS FORM $\begin{bmatrix} P_n \\ Q_n \end{bmatrix} = \sqrt{2}^{\,n} \cos\left(n\frac{\pi}{4}\right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \sqrt{2}^{\,n} \sin\left(n\frac{\pi}{4}\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$n=0$: $\begin{bmatrix} 8000 \\ 2000 \end{bmatrix} = \sqrt{2}^{\,0} \cos\left(0\frac{\pi}{4}\right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \sqrt{2}^{\,0} \sin\left(0\frac{\pi}{4}\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

AND $\begin{bmatrix} P_1 \\ Q_1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 8000 \\ 2000 \end{bmatrix} = \begin{bmatrix} 6000 \\ 10{,}000 \end{bmatrix}$

$n=1$: $\begin{bmatrix} 6000 \\ 10{,}000 \end{bmatrix} = \underbrace{\sqrt{2}^{\,1} \cos\left(1\frac{\pi}{4}\right)}_{1} \begin{bmatrix} 8000 \\ 2000 \end{bmatrix} + \underbrace{\sqrt{2}^{\,1} \sin\left(1\frac{\pi}{4}\right)}_{1} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$\begin{bmatrix} 6000 \\ 10{,}000 \end{bmatrix} = \begin{bmatrix} 8000 \\ 2000 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} -2000 \\ 8000 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$\therefore$ EXACT SOLUTION: $\boxed{\begin{bmatrix} P_n \\ Q_n \end{bmatrix} = \sqrt{2}^{\,n} \cos\left(n\frac{\pi}{4}\right) \begin{bmatrix} 8000 \\ 2000 \end{bmatrix} + \sqrt{2}^{\,n} \sin\left(n\frac{\pi}{4}\right) \begin{bmatrix} -2000 \\ 8000 \end{bmatrix}}$

6. (15 points) Find the fixed point of the system

$$P_{n+1} = P_n + 1.5Q_n - 20$$
$$Q_{n+1} = -0.75P_n + Q_n + 6$$

and determine if it is a sink, source, or saddle.

FIXED POINT: $\begin{matrix} P = P + 1.5Q - 20 \\ Q = -.75P + Q + 6 \end{matrix} \implies \begin{matrix} 1.5Q = 20 \\ .75P = 6 \end{matrix} \implies \begin{matrix} Q = 40/3 \\ P = 8 \end{matrix}$

$$\boxed{\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 8 \\ 40/3 \end{bmatrix}}$$

EIGENVALUES: $\det \begin{bmatrix} 1-\lambda & 1.5 \\ -.75 & 1-\lambda \end{bmatrix} = 0 \implies (1-\lambda)^2 + 9/8 = 0$

$(1-\lambda)^2 = -9/8 \implies 1-\lambda = \pm\sqrt{\frac{9}{8}}\, i$

$\lambda = 1 \pm \sqrt{\frac{9}{8}}\, i$ (COMPLEX CONJUGATES WITH SAME MODULUS/ABS. VALUE)

SINCE $|\lambda| = \sqrt{1^2 + \sqrt{\frac{9}{8}}^2} = \sqrt{\frac{17}{8}} > 1$,

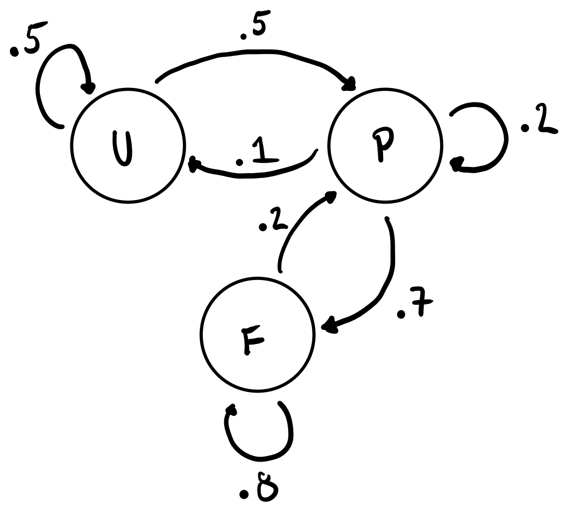
FIXED POINT IS A $\boxed{\text{SOURCE}}$

7. (15 points) Suppose trainers of guide dogs classify their subjects as being either untrained, partially trained or fully trained. Each month 50% of untrained dogs become partially trained, but the rest remain untrained; 20% of partially trained dogs remain partially trained, another 70% become fully trained, but 10% revert to untrained status; 80% of fully trained dogs remain fully trained, but 20% revert to partially trained status. What percent of these dogs will eventually be in each of these three classifications?

LET U_n = PROPORTION OF DOGS UNTRAINED AFTER n MONTHS

P_n = PROPORTION OF DOGS PARTIALLY TRAINED AFTER n MONTHS

F_n = PROPORTION OF DOGS FULLY TRAINED AFTER n MONTHS



WE HAVE

$$U_{n+1} = .5\,U_n + .1\,P_n$$
$$P_{n+1} = .5\,U_n + .2\,P_n + .2\,F_n$$
$$F_{n+1} = .7\,P_n + .8\,F_n$$

AND $U_n + P_n + F_n = 1$

$\Rightarrow F_n = 1 - U_n - P_n$

$$\therefore \quad U_{n+1} = .5\,U_n + .1\,P_n$$
$$P_{n+1} = .5\,U_n + .2\,P_n + .2\left(1 - U_n - P_n\right) = .3\,U_n + .2$$

THAT IS, $U_{n+1} = .5\,U_n + .1\,P_n$

$P_{n+1} = .3\,U_n + 0\,P_n + .2$

OR $\begin{bmatrix} U_{n+1} \\ P_{n+1} \end{bmatrix} = \begin{bmatrix} .5 & .1 \\ .3 & 0 \end{bmatrix} \begin{bmatrix} U_n \\ P_n \end{bmatrix} + \begin{bmatrix} 0 \\ .2 \end{bmatrix}$

EIGENVALUES: $\det \begin{bmatrix} .5-\lambda & .1 \\ .3 & -\lambda \end{bmatrix} = 0$

$$(.5-\lambda)(-\lambda) - .03 = 0$$
$$\lambda^2 - .5\lambda - .03 = 0$$
$$\lambda = \frac{.5 \pm \sqrt{.25 + .12}}{2} = .25 \pm .5\sqrt{.37}$$

$\lambda_1 \approx .55 \quad , \lambda_2 \approx -.05 \quad \Rightarrow \quad \boxed{\text{FIXED POINT IS A SINK (STABLE!)}}$

FIXED POINT:

$$U = .5U + .1P \qquad\qquad -.5U + .1P = 0 \quad ①$$
$$\Rightarrow$$
$$P = .3U + .2 \qquad\qquad -.3U + P = .2 \quad ②$$

$$\begin{array}{rrcl} -10\,①: & 5U - P & = & 0 \\ + \quad ②: & -.3U + P & = & .2 \\ \hline & 4.7U & = & .2 \end{array}$$

$$U = \frac{2}{47} \approx .0426 \quad , \quad P = .3U + .2 \approx .2128$$

$$F = 1 - U - P \approx .7446$$

FIXED POINT:

$$\boxed{\begin{array}{l} U \approx 4.26\% \\ P \approx 21.28\% \\ F \approx 74.46\% \end{array}}$$