Written Homework 6

Sections 4.1-4

Due 12/1

1. Construct the linear competition model with no immigration, migration, or harvesting that satisfies:

$$P_0 = 100 \quad P_1 = 150 \quad P_2 = 250$$
$$Q_0 = 200 \quad Q_1 = 100 \quad Q_2 = 25$$

MODEL $P_{n+1} = r_1 P_n - s_1 Q_n$
$Q_{n+1} = -s_2 P_n + r_2 Q_n$

$n=0$: $150 = 100 r_1 - 200 s_1$
$100 = -100 s_2 + 200 r_2$

$n=1$: $250 = 150 r_1 - 100 s_1$
$25 = -150 s_2 + 100 r_2$

$\Rightarrow$ 2 SYSTEMS OF EQUATIONS: $150 = 100 r_1 - 200 s_1$, $250 = 150 r_1 - 100 s_1$ and $100 = -100 s_2 + 200 r_2$, $25 = -150 s_2 + 100 r_2$

$350 = 200 r_1$
$r_1 = 7/4$, $s_1 = 1/8$

$50 = 200 s_2$
$s_2 = 1/4$, $r_2 = 5/8$

$\therefore$ MODEL: $\boxed{\begin{aligned} P_{n+1} &= \tfrac{7}{4} P_n - \tfrac{1}{8} Q_n \\ Q_{n+1} &= -\tfrac{1}{4} P_n + \tfrac{5}{8} Q_n \end{aligned}}$

2. Find the unique fixed point of the following linear system.

$$P_{n+1} = 4P_n - Q_n + 40$$
$$Q_{n+1} = .75P_n + .5Q_n + 20$$

FIXED POINT: $P_n = P$, $Q_n = Q$ $\forall n \geq 0$.

$P = 4P - Q + 40$
$Q = .75P + .5Q + 20$

$\Rightarrow$ $3P - Q = -40$
$.75P - .5Q = -20$ $\quad -1.5P + Q = 40$

$1.5P = 0 \Rightarrow \boxed{P = 0, \ Q = 40}$

3. Let A and B be the matrices

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 4 \\ 3 & 1 \end{bmatrix},$$

and let $\vec{u}$ and $\vec{v}$ be the vectors

$$\vec{u} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

(a) Show by computation that $A(4\vec{u} - 5\vec{v}) = 4A\vec{u} - 5A\vec{v}$.

(b) Show by computation that $(A + B)\vec{u} = A\vec{u} + B\vec{u}$.

(c) Show by computation that $(AB)^{-1} = B^{-1}A^{-1}$.

(a) $A(4\vec{u} - 5\vec{v}) = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}\left(4\begin{bmatrix} -3 \\ 2 \end{bmatrix} - 5\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}\left(\begin{bmatrix} -12 \\ 8 \end{bmatrix} - \begin{bmatrix} 5 \\ -10 \end{bmatrix}\right)$

$= \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -17 \\ 18 \end{bmatrix} = \begin{bmatrix} -69 \\ 19 \end{bmatrix}$

$4A\vec{u} - 5A\vec{v} = 4\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix} - 5\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 4\begin{bmatrix} -11 \\ 1 \end{bmatrix} - 5\begin{bmatrix} 5 \\ -3 \end{bmatrix}$

$= \begin{bmatrix} -44 \\ 4 \end{bmatrix} - \begin{bmatrix} 25 \\ -15 \end{bmatrix} = \begin{bmatrix} -69 \\ 19 \end{bmatrix}$ ✓

(b) $(A+B)\vec{u} = \left(\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 3 & 1 \end{bmatrix}\right)\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$

$A\vec{u} + B\vec{u} = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -11 \\ 1 \end{bmatrix} + \begin{bmatrix} 14 \\ -7 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$ ✓

(c) $(AB)^{-1} = \left(\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 3 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} -9 & 11 \\ 4 & 6 \end{bmatrix}^{-1} = \frac{-1}{98}\begin{bmatrix} 6 & -11 \\ -4 & -9 \end{bmatrix}$

$B^{-1}A^{-1} = \begin{bmatrix} -2 & 4 \\ 3 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}^{-1} = -\frac{1}{14}\begin{bmatrix} 1 & -4 \\ -3 & -2 \end{bmatrix}\frac{1}{7}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \frac{-1}{98}\begin{bmatrix} 6 & -11 \\ -4 & -9 \end{bmatrix}$ ✓

4. Write the system

$$3x - 6y = 0$$
$$5x - 9y = 1$$

in the vector/matrix form $A\vec{x} = \vec{b}$. Then solve for x and y using $\vec{x} = A^{-1}\vec{b}$.

$\begin{bmatrix} 3 & -6 \\ 5 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 5 & -9 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -9 & 6 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \boxed{\begin{matrix} x = 2 \\ y = 1 \end{matrix}}$

5. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 6 & 1 \\ -2 & 3 \end{bmatrix}$$

and find the set of eigenvectors corresponding to each eigenvalue.
Remember that eigenvectors are not unique. If $\vec{v}$ is an eigenvector then so is $c\vec{v}$ for all nonzero real numbers c.

CHARACTERISTIC EQUATION: $\det(A - rI) = 0$

$$\det\left(\begin{bmatrix} 6 & 1 \\ -2 & 3 \end{bmatrix} - \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\right) = \det\begin{bmatrix} 6-r & 1 \\ -2 & 3-r \end{bmatrix} = (6-r)(3-r)+2 = 0$$

$$18 - 6r - 3r + r^2 + 2$$

$$r^2 - 9r + 20 = 0$$

$$(r-5)(r-4) = 0 \Rightarrow r = 5, 4 \quad \text{EIGENVALUES.}$$

$$r = 5: \quad A\vec{v} = r\vec{v} \Rightarrow (A - rI)\vec{v} = \vec{0} \Rightarrow \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0}$$

$$\Rightarrow \left.\begin{array}{l} v_1 + v_2 = 0 \\ -2v_1 - 2v_2 = 0 \end{array}\right\} \text{EQUIVALENT (OF COURSE)} \Rightarrow v_1 = -v_2 \quad \therefore$$

$$\boxed{\text{EIGENVALUE } r = 5 \text{ HAS SET OF EIGENVECTORS } c\begin{bmatrix} 1 \\ -1 \end{bmatrix},\ c \neq 0.}$$

$$r = 4: \quad (A - 4I)\vec{v} = 0 \Rightarrow \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0} \Rightarrow \begin{array}{l} 2v_1 + v_2 = 0 \\ -2v_1 - v_2 = 0 \end{array}$$

$$\Rightarrow v_2 = -2v_1 \quad \therefore$$

$$\boxed{\text{EIGENVALUE } r = 4 \text{ HAS SET OF EIGENVECTORS } c\begin{bmatrix} 1 \\ -2 \end{bmatrix},\ c \neq 0.}$$

6. Determine whether the origin is a sink, source, or saddle for the linear system

$$\vec{x}_{n+1} = A\vec{x}_n.$$

(a) $A = \begin{bmatrix} 1/4 & -1/8 \\ -1/2 & 3/4 \end{bmatrix}$

FIND EIGENVALUES OF A:

$$\det\begin{bmatrix} 1/4 - r & -1/8 \\ -1/2 & 3/4 - r \end{bmatrix} = \left(\tfrac{1}{4} - r\right)\left(\tfrac{3}{4} - r\right) - \tfrac{1}{16} = 0$$

$$\tfrac{3}{16} - \tfrac{1}{4}r - \tfrac{3}{4}r + r^2 - \tfrac{1}{16} = 0 \Rightarrow r^2 - r + \tfrac{1}{8} = 0$$

$$\left(r - \tfrac{1}{2}\right)^2 - \tfrac{1}{8} = 0 \Rightarrow r - \tfrac{1}{2} = \pm\sqrt{1/8} \Rightarrow r = \tfrac{1}{2} \pm \sqrt{\tfrac{1}{8}}$$

NOTE: $\tfrac{1}{8} < \tfrac{1}{4} \Rightarrow \sqrt{1/8} < \tfrac{1}{2}$

$$\boxed{\begin{array}{l} \text{ABSOLUTE VALUE OF BOTH (REAL, DISTINCT) EIGENVALUES IS LESS THAN } 1. \\ \therefore \text{ THE FIXED POINT (ORIGIN) IS A SINK.} \end{array}}$$

(b) $A = \begin{bmatrix} 2.25 & 2.5 \\ 0.75 & 2 \end{bmatrix}$

$$\text{DET} \begin{bmatrix} 2.25 - r & 2.5 \\ .75 & 2 \end{bmatrix} = (2.25 - r)(2 - r) - 1.875$$

$$4.5 - 2.25r - 2r + r^2 - 1.875 = r^2 - 4.25r + 2.625 = 0$$

$$8r^2 - 34r + 21 = 0$$
$$(4r - 3)(2r - 7) = 0 \implies r = 3/4 ,\ 7/2$$

ABSOLUTE VALUE OF ONE EIGENVALUE IS LESS THAN 1,
ABSOLUTE VALUE OF THE OTHER EIGENVALUE IS GREATER THAN 1.
$\therefore$ THE FIXED POINT (ORIGIN) IS A SADDLE.