Homework 4

Sections 3.1-3.3

Solutions

1. Find the logistic model

$$P_{n+1} = rP_n\left(1 - \frac{P_n}{C}\right)$$

that satisfies the following conditions.

6 points (a) $P_0 = 5000, P_1 = 8000, P_2 = 6000.$

$$P_1 = rP_0\left(1 - \frac{P_0}{C}\right) \Rightarrow 8000 = r(5000)\left(1 - \frac{5000}{C}\right) \Rightarrow \frac{8}{5} = r\left(1 - \frac{5000}{C}\right) \quad (1)$$

$$P_2 = rP_1\left(1 - \frac{P_1}{C}\right) \Rightarrow 6000 = r(8000)\left(1 - \frac{8000}{C}\right) \Rightarrow \frac{6}{8} = r\left(1 - \frac{8000}{C}\right) \quad (2)$$

$$(1) \div (2): \quad \frac{8/5}{6/8} = \frac{1 - \frac{5000}{C}}{1 - \frac{8000}{C}} \Rightarrow \frac{64}{30} = \frac{C - 5000}{C - 8000}$$

$$\Rightarrow 64(C - 8000) = 30(C - 5000) \Rightarrow 64C - 512{,}000 = 30C - 150{,}000$$

$$\Rightarrow 34C = 362{,}000 \Rightarrow C = \frac{362{,}000}{34} = \frac{181{,}000}{17}$$

$$\text{Then } (1) \Rightarrow r = \frac{8}{5\left(1 - \frac{5000}{C}\right)} = \frac{8}{5\left(1 - \frac{5000(17)}{181000}\right)} = \frac{8(181)}{5(181 - 5(17))} = \frac{181}{60}$$

$$\therefore \boxed{P_{n+1} = \frac{181}{60} P_n \left(1 - \frac{17P_n}{181{,}000}\right)} \quad \text{or} \quad P_{n+1} = 3.0167 P_n\left(1 - \frac{P_n}{10647.0588}\right)$$

6 points (b) If $P_0 = 5000$ then P_1 is twice P_0, and if $P_0 = 8000$ then P_1 is half P_0.

$$10{,}000 = 5000r\left(1 - \frac{5000}{C}\right) \quad (1)$$

$$4000 = 8000r\left(1 - \frac{8000}{C}\right) \quad (2)$$

$$(1) \div (2) \Rightarrow \quad \frac{10}{4} = \frac{5\left(1 - \frac{5000}{C}\right)}{8\left(1 - \frac{8000}{C}\right)} = \frac{5C - 25{,}000}{8C - 64{,}000}$$

$$\Rightarrow 10(8C - 64000) = 4(5C - 25{,}000)$$

$$40C - 320{,}000 = 10C - 50{,}000$$

$$30C = 270{,}000 \Rightarrow C = 9{,}000$$

$$r = \frac{10{,}000}{5000\left(1 - \frac{5000}{9000}\right)} = \frac{2}{1 - 5/9} = \frac{18}{4} = \frac{9}{2}$$

$$\therefore \boxed{P_{n+1} = \frac{9}{2} P_n \left(1 - \frac{P_n}{9000}\right)}$$

2. Find Ricker model

$$P_{n+1} = rP_n e^{-P_n/N}$$

that satisfies the following conditions.

6 points (a) $N = 1000, P_0 = 100, P_1 = 250.$

$$250 = r(100)e^{-100/1000} \implies r = \frac{5}{2}e^{1/10}$$

$$\therefore \boxed{P_{n+1} = \frac{5}{2}e^{1/10} P_n e^{-P_n/1000}} \quad \text{or} \quad P_{n+1} = \frac{5}{2}P_n e^{\frac{1}{10} - \frac{P_n}{1000}}$$

$$\text{or} \quad P_{n+1} = 2.7629\, P_n e^{-P_n/1000}$$

6 points (b) $P_0 = 2000, P_1 = 6000, P_2 = 4000.$

$$\left.\begin{array}{l} 6000 = 2000\, r e^{-2000/N} \\ 4000 = 6000\, r e^{-6000/N} \end{array}\right\} \implies \frac{6}{4} = \frac{2}{6} e^{\frac{6000-2000}{N}}$$

$$\implies \frac{9}{2} = e^{4000/N} \implies \frac{4000}{N} = \ln\left(\frac{9}{2}\right)$$

$$\implies N = \frac{4000}{\ln\left(\frac{9}{2}\right)} \implies e^{-P_n/N} = e^{\frac{-P_n \ln\left(\frac{9}{2}\right)}{4000}} = \left(\frac{9}{2}\right)^{-P_n/4000} = \left(\frac{2}{9}\right)^{P_n/4000}$$

$$\implies r = \frac{6000\, e^{\frac{2000\ln(9/2)}{4000}}}{2000} = 3\left(\frac{9}{2}\right)^{1/2} \implies r = \frac{9}{\sqrt{2}}$$

$$\text{Note: } e^{\frac{2000\ln(9/2)}{4000}} = \left[e^{\ln\left(\frac{9}{2}\right)}\right]^{\frac{2000}{4000}} = \left(\frac{9}{2}\right)^{1/2} \qquad \therefore \boxed{P_{n+1} = \frac{9}{\sqrt{2}} P_n \left(\frac{2}{9}\right)^{P_n/4000}}$$

$$\text{or} \quad P_{n+1} = 6.3640\, P_n e^{-P_n/44.4360}$$

3. The formula for the exact solution of $x_{n+1} = (x_n - c)^a + c$, where c is a constant, can be derived as follows"

2 points

(a) Use the substitution $u_n = x_n - c$ to convert $x_{n+1} = (x_n - c)^a + c$ into $u_{n+1} = u_n^a$.

$u_n = x_n - c$ For all n $\Rightarrow$ $u_{n+1} = x_{n+1} - c$

$$x_{n+1} = (x_n - c)^a + c \Rightarrow \underbrace{x_{n+1} - c}_{\downarrow} = (\underbrace{x_n - c}_{\downarrow})^a$$

$$u_{n+1} = u_n^a$$

✓

4 points

(b) Find the exact solution for u_n.

$u_1 = u_0^a$

$u_2 = u_1^a = (u_0^a)^a = u_0^{a^2}$

$u_3 = u_2^a = (u_0^{a^2})^a = u_0^{a^3}$

$\dots \quad \boxed{u_n = u_0^{a^n}}$

2 points

(c) Use the substitution $u_n = x_n - c$ again to find an exact solution for x_n.

$x_n - c = (x_0 - c)^{a^n}$

$$\boxed{x_n = (x_0 - c)^{a^n} + c}$$

4. For each of the following equations, find all fixed points and use a substitution of the type in question 3 to find an exact solution. Then use the exact solution to classify each fixed point as either stable or unstable, and find the basin of attraction for all stable fixed points.

10 POINTS (a) $x_{n+1} = 5 + \sqrt{x_n - 5}$

FIXED POINT(S): $f(p) = p \Rightarrow 5 + \sqrt{p-5} = p \Rightarrow p - 5 = (p-5)^2$

$p = 5$ OR 6

EXACT SOL'N: $x_n = (x_0 - 5)^{1/2^n} + 5$

IF $x_0 < 5$ THEN x_1 IS UNDEFINED.

IF $5 < x_0 \neq 6$ THEN $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} (x_0-5)^{1/2^n} + 5 = 1 + 5 = 6$

$\therefore$ $p = 5$ IS UNSTABLE,
$p = 6$ IS STABLE WITH BASIN OF ATTRACTION $(5, \infty)$.

10 POINTS (b) $x_{n+1} = x_n^2 - 2x_n + 2$

$\hookrightarrow x_{n+1} - 1 = (x_n - 1)^2$

FIXED POINTS: $p = 1, 2$

EXACT SOLUTION: $x_n = (x_0 - 1)^{2^n} + 1$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (x_0-1)^{2^n} + 1 = \begin{cases} 1 & \text{IF } |x_0 - 1| < 1 \\ 2 & \text{IF } |x_0 - 1| = 1 \\ \infty & \text{IF } |x_0 - 1| > 1 \end{cases}$$

$\therefore$ $p = 1$ IS STABLE WITH BASIN OF ATTRACTION $(0, 2)$
$p = 2$ IS UNSTABLE

5. For each of the following functions, find all fixed points (if any) algebraically, and use the derivative to classify each as either stable or unstable.

8 POINTS (a) $f(x) = 4x^5$

$4p^5 = p$

$p(4p^4 - 1) = 0$

$p(2p^2+1)(\sqrt{2}p+1)(\sqrt{2}p-1) = 0$

$\Rightarrow p = 0, \pm\frac{1}{\sqrt{2}}$

$f'(x) = 20x^4$

$f'(0) = 0$

SINCE $|f'(0)| < 1$, $p = 0$ IS STABLE

$f'\left(\pm\frac{1}{\sqrt{2}}\right) = 20\left(\frac{1}{4}\right) = 5$

SINCE $\left|f'\left(\pm\frac{1}{2}\right)\right| > 1$, $p = \pm\frac{1}{\sqrt{2}}$ ARE UNSTABLE

8 POINTS (b) $f(x) = 4xe^{-x}$

$4pe^{-p} = p$

$p(4e^{-p} - 1) = 0$

$p = 0$ OR $e^{-p} = \frac{1}{4}$

$p = \ln 4$

$f'(x) = 4e^{-x}(1-x)$

$f'(0) = 4 > 1$

$f'(\ln 4) = 1 - \ln 4$ BETWEEN -1 & 0

$\begin{pmatrix} \ln e < \ln 4 < \ln e^2 \\ 1 < \ln 4 < 2 \end{pmatrix}$

$\therefore$ $\boxed{\begin{array}{l} p = 0 \text{ IS UNSTABLE} \\ p = \ln 4 \text{ IS STABLE} \end{array}}$

8 POINTS (c) $f(x) = \dfrac{2}{x+1}$

$\frac{2}{p+1} = p$

$0 = p^2 + p - 2$

$0 = (p+2)(p-1)$

$\therefore p = -2, 1$

$f'(x) = \frac{-2}{(x+1)^2}$

$f'(-2) = \frac{-2}{(-2+1)^2} = -2 < -1$

$f'(1) = \frac{-2}{(1+1)^2} = -\frac{1}{2}$ BETWEEN -1 & 0

$\therefore$ $\boxed{\begin{array}{l} p = -2 \text{ IS UNSTABLE} \\ p = 1 \text{ IS STABLE} \end{array}}$

8 POINTS (d) $f(x) = x^{1/4}$, $x_0 > 0$

$p^{1/4} = p$

$p = p^4$

$p(p^3 - 1) = 0$

$\underbrace{p = 0}_{\text{NOT POSITIVE} \Rightarrow \text{IGNORE!}}$ OR $p = 1$

$f'(x) = \frac{1}{4}x^{-3/4}$

$f'(1) = \frac{1}{4}$ BETWEEN 0 & 1

$\therefore$ $\boxed{p = 1 \text{ IS STABLE}}$

6. Consider the function

$$f(x) = \frac{3x-1}{2x^2-x+1}$$

4 POINTS

(a) Use the graph below to identify all fixed points of f and use algebra to confirm your answer.

$$\frac{3x-1}{2x^2-x+1} = x \quad \Rightarrow \quad 3x-1 = 2x^3 - x^2 + x$$

$$0 = 2x^3 - x^2 - 2x + 1 = x^2(2x-1) - (2x-1) = (2x-1)(x^2-1)$$

$$0 = (2x-1)(x+1)(x-1)$$

FIXED POINTS : $\boxed{x = -1,\ \tfrac{1}{2},\ 1}$

X- COORD. OF INTERSECTIONS OF $y = f(x)$ & $y = x$.

6 POINTS

(b) Create some cobweb graphs to determine the stability of each fixed point and use f' to confirm your answer.

$$f'(x) = \frac{(2x^2-x+1)(3) - (3x-1)(4x-1)}{(2x^2-x+1)^2}$$

$$f'(-1) = \frac{(2+1+1)(3) - (-3-1)(-4-1)}{(2+1+1)^2} = \frac{12-20}{16} = -\frac{1}{2} \qquad |f'(-1)| < 1 \Rightarrow x = -1 \text{ IS STABLE}$$

$$f'(\tfrac{1}{2}) = \frac{(\frac{1}{2}-\frac{1}{2}+1)(3) - (\frac{3}{2}-1)(2-1)}{(\frac{1}{2}-\frac{1}{2}+1)^2} = \frac{3-\frac{1}{2}}{1} = \frac{5}{2} \qquad |f'(\tfrac{1}{2})| > 1 \Rightarrow x = \tfrac{1}{2} \text{ IS UNSTABLE}$$

$$f'(1) = \frac{(2-1+1)(3) - (3-1)(4-1)}{(2-1+1)^2} = \frac{6-6}{4} = 0 \qquad |f'(1)| < 1 \Rightarrow x = 1 \text{ IS STABLE}$$

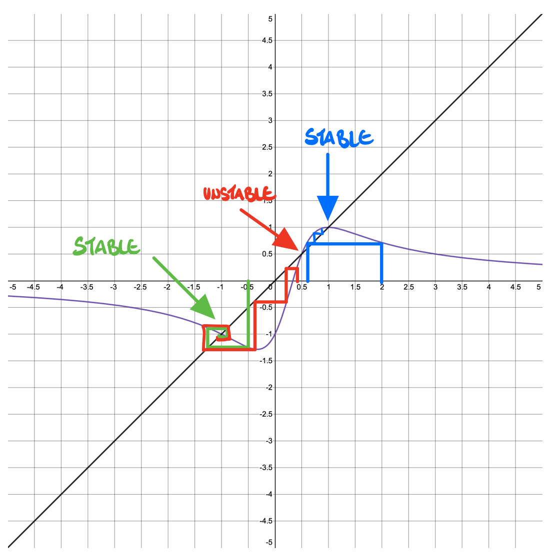

Figure 1: https://www.desmos.com/calculator/hpaczexfzq

6 POINTS

(c) For each stable fixed point (if any), use cobweb graphs to determine its basin of attraction, i.e., the largest interval containing the stable fixed point p in which solutions converge to p.

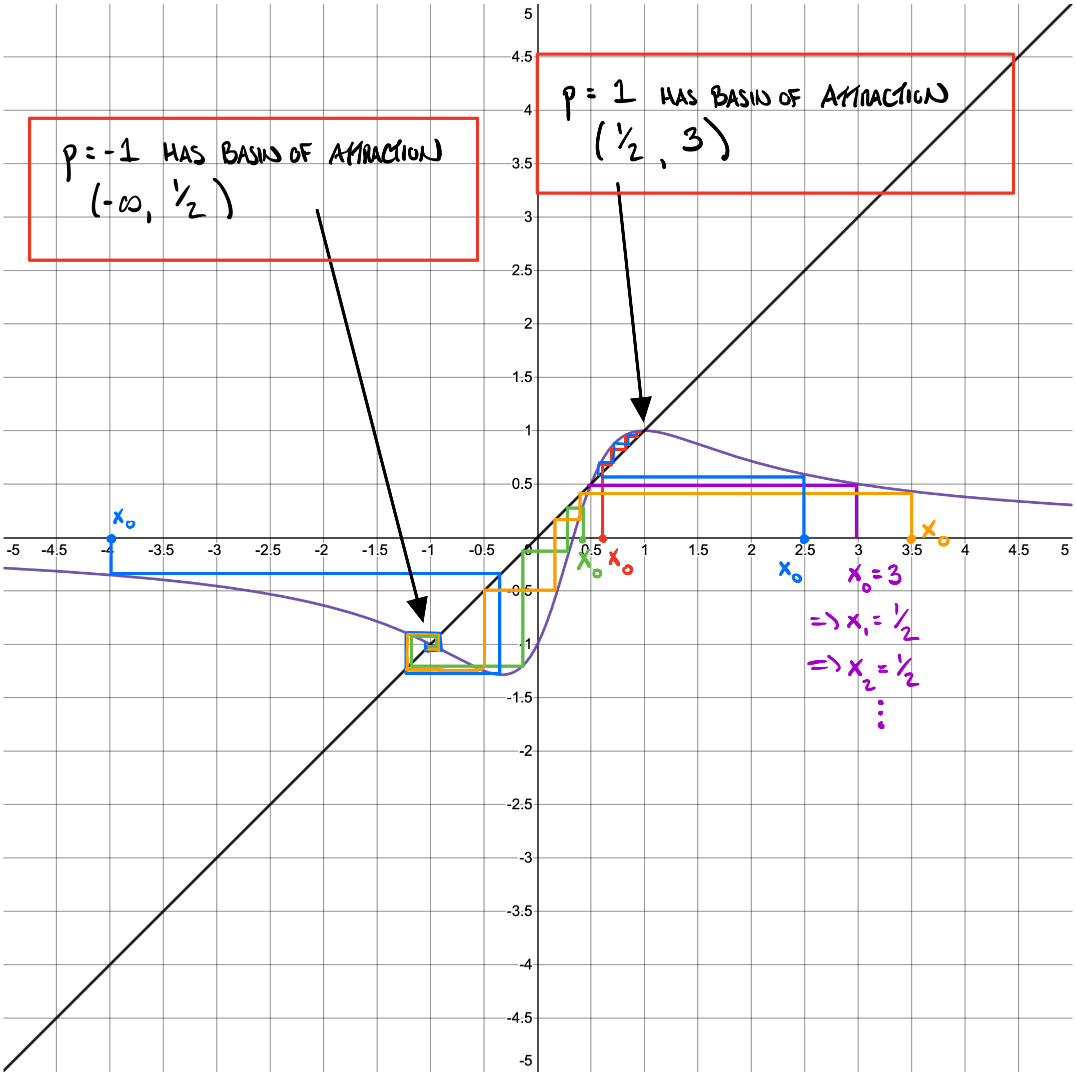

Figure 2: https://www.desmos.com/calculator/hpaczexfzq