# Homework 3

§2.6-7

1. (8 points) Find the fixed point of the following linear model.

$$x_{n+1} = (7 - 3x_n)/2$$

   Cassify the fixed point as either stable or unstable, and state whether or not solutions (other than the fixed point) oscillate.

2. (8 points) Find the equilibrium (fixed point) of the population model

$$P_{n+1} = 0.95P_n + 7.25$$

   Determine both stability and whether or not oscillation occurs.

3. Suppose that each day during flu season, 15% of those who have the flu in a certain town recover from it, while another 600 people come down with the flu. If there are currently estimated to be 1800 cases of the flu:
   (a) (8 points) How many cases will there be two weeks (14 days) from now?
   (b) (6 points) Does the number of cases eventually stabilize? If so, at what number?
   (c) (8 points) Suppose the estimated number of current cases is wildly inaccurate. If there are actually 10,000 current cases of the flu, does the answer to part (b) change?

4. Find the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ for each of the following functions.
   (a) (8 points) $f(x, y) = 7x^3 + 6x^2y + y^3$
   (b) (8 points) $f(x, y) = \sin(xy^2)$

5. (10 points) Let

$$f(x, y) = x^2 + 2xy + 4y^2 - 3y.$$

   Find the critical point(s) of $f$, i.e., the points $(x_0, y_0)$ that satisfy both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.

6. (12 points) Consider the following set of 5 points.

$$\{(1, 7), (2, 10), (3, 11), (4, 14), (6, 19)\}$$

   Construct the least-squares regression line that best approximates the points.

7. (12 points) Suppose the following data is collected through observation.

$$x_0 = 60, \quad x_1 = 25, \quad x_2 = 10, \quad x_3 = 5, \quad x_4 = 2, \quad x_5 = 1$$

   Construct the linear model $x_{n+1} = ax_n + b$ that best approximates the data.

8. (12 points) In the special case that a function $f$ is invertible, the inverse function $f^{-1}$ exists and the linear equation $x_{n+1} = f(x_n)$ can be iterated *backward* according to the equation

$$x_{n-1} = f^{-1}(x_n).$$

   Thus, given an initial value $x_0$, iteration of $f$ produces the *forward orbit*

$$x_0, x_1, x_2, \ldots$$

   and iteration of $f^{-1}$ produces the *backward orbit*

$$x_0, x_{-1}, x_{-2}, \ldots.$$

   Prove that a fixed point $p$ of a *linear* function $f(x) = ax + b$ is stable under forward iteration if and only if it is unstable under backward iteration.

# Homework 3

§2.6-7

1. (8 points) Find the fixed point of the following linear model.

$$x_{n+1} = (7 - 3x_n)/2$$

Cassify the fixed point as either stable or unstable, and state whether or not solutions (other than the fixed point) oscillate.

LET $p$ = FIXED POINT.

THEN $p = \dfrac{7-3p}{2} \Rightarrow 2p = 7 - 3p \Rightarrow p = \dfrac{7}{5}$ OR $1.4$

THE MODEL IS LINEAR $x_{n+1} = -\frac{3}{2}x_n + \frac{7}{2}$

AND THE COEFFICIENT OF $x_n$, $-\frac{3}{2}$.

$\left|-\frac{3}{2}\right| > 1 \Rightarrow$ $p$ IS UNSTABLE

$-\frac{3}{2} < 0 \Rightarrow$ SOLUTIONS OSCILLATE

2. (8 points) Find the equilibrium (fixed point) of the population model

$$P_{n+1} = 0.95P_n + 7.25$$

Determine both stability and whether or not oscillation occurs.

LET $p$ = FIXED POINT.

$p = .95p + 7.25 \Rightarrow .05p = 7.25 \Rightarrow p = 145$

$|.95| < 1 \Rightarrow$ $p$ IS STABLE

$.95 > 1 \Rightarrow$ SOLUTIONS DO NOT OSCILLATE

(i.e. SOLUTIONS ARE MONOTONIC)

3. Suppose that each day during flu season, 15% of those who have the flu in a certain town recover from it, while another 600 people come down with the flu. If there are currently estimated to be 1800 cases of the flu:

   (a) (8 points) How many cases will there be two weeks (14 days) from now?

   (b) (6 points) Does the number of cases eventually stabilize? If so, at what number?

   (c) (8 points) Suppose the estimated number of current cases is wildly inaccurate. If there are actually 10,000 current cases of the flu, does the answer to part (b) change?

(a) LET $F_n$ = # FLU CASES $n$ WEEKS FROM NOW.

$$F_{n+1} = .85 F_n + 600, \quad F_0 = 1800$$

$$\therefore F_n = .85^n (1800) + \frac{600(1-.85^n)}{1-.85}$$

$$F_{14} = .85^{14}(1800) + \frac{600(1-.85^{14})}{1-.85} \approx 3{,}774 \text{ CASES}$$

(b) YES, SINCE $|.85| < 1$, THE FIXED POINT IS STABLE AND ALL SOLUTIONS CONVERGE TO THAT VALUE:

$$p = .85p + 600 \Rightarrow .15p = 600 \Rightarrow p = 4000$$

(c) NO. ALL SOLUTIONS CONVERGE TO THE FIXED POINT $p = 4000$.

4. Find the partial derivatives $f_x(x,y)$ and $f_y(x,y)$ for each of the following functions.

   (a) (8 points) $f(x,y) = 7x^3 + 6x^2y + y^3$

   (b) (8 points) $f(x,y) = \sin(xy^2)$

(a) $f_x = 21x^2 + 12xy$ $\qquad f_y = 6x^2 + 3y^2$

(b) $f_x = y^2 \cos(xy^2)$ $\qquad f_y = 2xy \cos(xy^2)$

5. (10 points) Let

$$f(x,y) = x^2 + 2xy + 4y^2 - 3y.$$

Find the critical point(s) of $f$, i.e., the points $(x_0, y_0)$ that satisfy both $f_x(x_0,y_0) = 0$ and $f_y(x_0,y_0) = 0$.

$f_x = 2x + 2y$ SOLVE THE SYSTEM OF EQ'S $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$

$f_y = 2x + 8y - 3$

$\Rightarrow \quad 2x_0 + 2y_0 = 0 \qquad \Rightarrow 6y_0 = 3 \quad \Rightarrow y_0 = 1/2$

$\quad\;\; 2x_0 + 8y_0 = 3 \qquad\qquad\qquad\quad \Rightarrow x_0 = -1/2$

$\therefore$ CRITICAL POINT $(x_0, y_0) = (-1/2, 1/2)$

6. (12 points) Consider the following set of 5 points.

$$\{(1,7),(2,10),(3,11),(4,14),(6,19)\}$$

Construct the least-squares regression line that best approximates the points.

| | $x_i$ | $y_i$ | $x_i^2$ | $x_i y_i$ |
|---|---|---|---|---|
| | 1 | 7 | 1 | 7 |
| | 2 | 10 | 4 | 20 |
| | 3 | 11 | 9 | 33 |
| | 4 | 14 | 16 | 56 |
| | 6 | 19 | 36 | 114 |
| SUM | 16 | 61 | 66 | 230 |

NORMAL EQ'S: $\left(\sum x_i^2\right) m + \left(\sum x_i\right) b = \sum x_i y_i$

$\left(\sum x_i\right) m + Nb = \sum y_i$

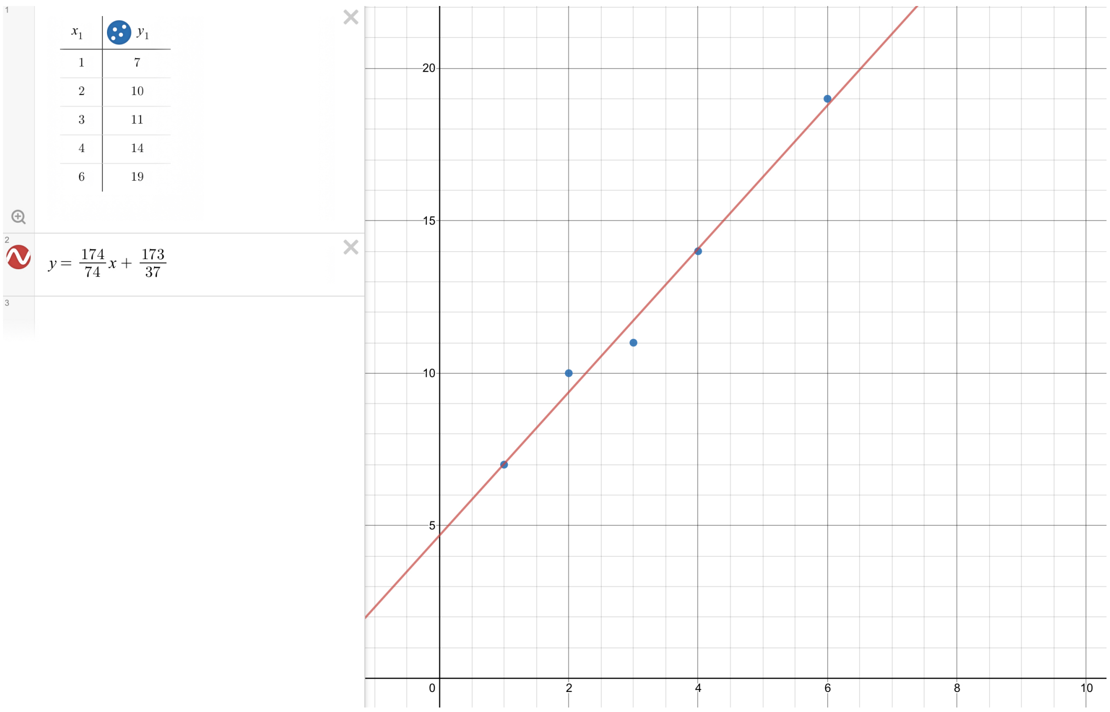
$\Rightarrow 66m + 16b = 230 \xrightarrow{\times 5} 330m + 80b = 1150$

$16m + 5b = 61 \xrightarrow{\times 16} 256m + 80b = 976$

$\Rightarrow 74m = 174 \Rightarrow m = \dfrac{174}{74}$,

$$b = \frac{61 - 16\left(\frac{174}{74}\right)}{5} = \frac{(61)(74) - (16)(174)}{(5)(74)} = \frac{1730}{370} = \frac{173}{37}$$

$\therefore \quad y = mx + b \longrightarrow y = \dfrac{174}{74}x + \dfrac{173}{37}$ OR $y = 2.35x + 4.68$




7. (12 points) Suppose the following data is collected through observation.

$$x_0 = 60, \quad x_1 = 25, \quad x_2 = 10, \quad x_3 = 5, \quad x_4 = 2, \quad x_5 = 1$$

Construct the linear model $x_{n+1} = ax_n + b$ that best approximates the data.

| | $x_i$ | $y_i$ | $x_i^2$ | $x_i y_i$ |
|---|---|---|---|---|
| | 60 | 25 | 3600 | 1500 |
| | 25 | 10 | 625 | 250 |
| | 10 | 5 | 100 | 50 |
| | 5 | 2 | 25 | 10 |
| | 2 | 1 | 4 | 2 |
| SUM | 102 | 43 | 4354 | 1812 |

NORMAL EQ'S:

$$\left(\sum x_i^2\right) m + \left(\sum x_i\right) b = \sum x_i y_i$$
$$\left(\sum x_i\right) m + N b = \sum y_i$$

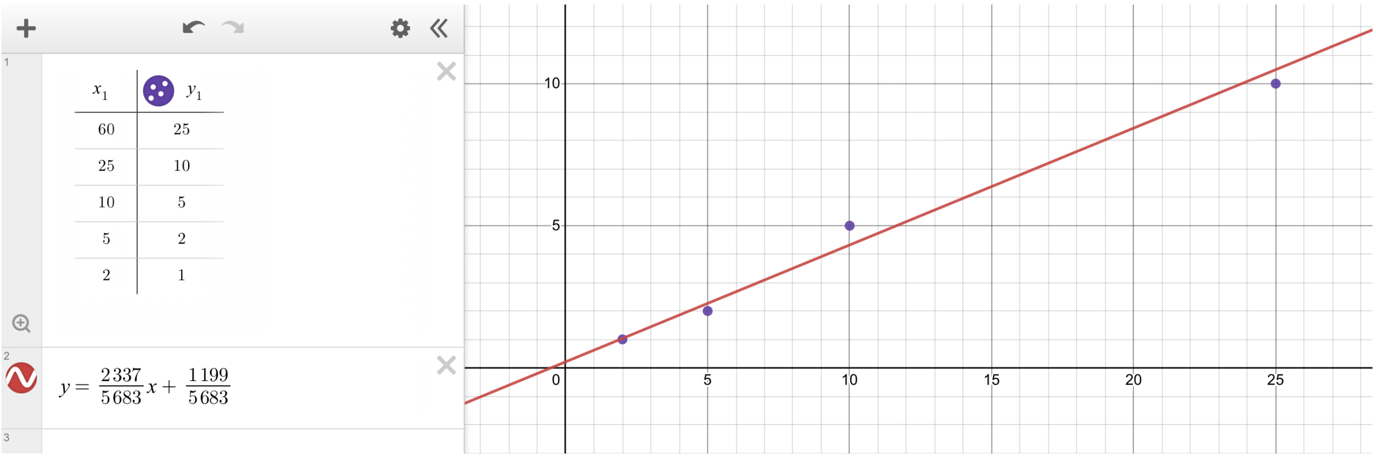
$$4354\, m + 102\, b = 1812$$
$$102\, m + 5\, b = 43$$

$$5\,(4354\, m + 102\, b = 1812) \rightarrow 21770\, m + 510\, b = 9060$$
$$-102\,(102\, m + 5\, b = 43) \rightarrow -10404\, m - 510\, b = -4386 \quad +$$
$$11366\, m = 4674$$

$$m = \frac{4674}{11366} = \frac{2337}{5683}$$

$$b = \frac{43 - 102m}{5} = \frac{43 - 102\left(\frac{2337}{5683}\right)}{5} = \frac{1199}{5683}$$

$\therefore$ BEST LINEAR MODEL: $x_{n+1} = \left(\frac{2337}{5683}\right) x_n + \frac{1199}{5683}$




https://www.desmos.com/calculator/d5lIndOeci

8. (12 points) In the special case that a function $f$ is invertible, the inverse function $f^{-1}$ exists and the linear equation $x_{n+1} = f(x_n)$ can be iterated *backward* according to the equation

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Prove that a fixed point $p$ of a *linear* function $f(x) = ax + b$ is stable under forward iteration if and only if it is unstable under backward iteration.

GIVEN $f(x) = ax + b$ LINEAR,

WE HAVE $f^{-1}(x) = \dfrac{x-b}{a} = \dfrac{1}{a}x - \dfrac{b}{a}$ ALSO LINEAR.

FIXED POINT OF $f$ IS $p = \dfrac{b}{1-a}$.

FIXED POINT OF $f^{-1}$ IS $p = \dfrac{-b/a}{1 - 1/a} = \dfrac{b}{1-a}$

SAME ✓

•) IF $p$ IS A STABLE FIXED POINT OF $f$ THEN $|a| < 1$.
THIS IMPLIES $\left|\frac{1}{a}\right| > 1$ AND SO $p$ IS AN UNSTABLE FIXED POINT OF $f^{-1}$.

•) ON THE OTHER HAND, IF $p$ IS AN UNSTABLE FIXED POINT OF $f$ THEN $|a| > 1$.
THIS IMPLIES $\left|\frac{1}{a}\right| < 1$ AND SO $p$ IS A STABLE FIXED POINT OF $f^{-1}$.

$\square$