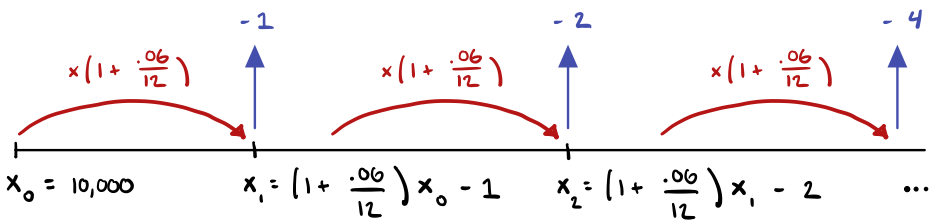
# Homework 2

Solutions

6 Points

1. Suppose someone borrows \$10,000 at an annual interest rate of 6% compounded monthly and makes monthly payments on the loan. Construct an iterative model $x_{n+1} = f(x_n)$ for the monthly unpaid balance if the monthly payments are \$1, \$2, \$4, \$8, \$16, $\cdots$.

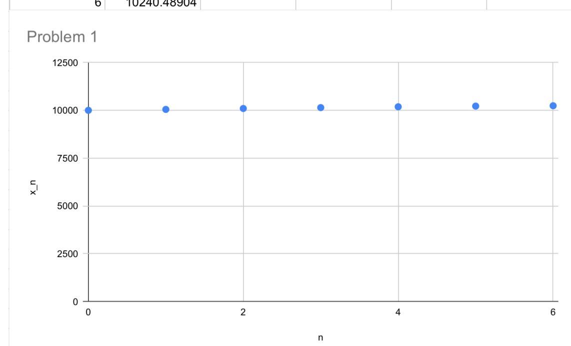



$$x_{n+1} = \left(1 + \frac{.06}{12}\right) x_n - 2^n$$

$$\boxed{x_{n+1} = 1.005\, x_n - 2^n}$$

$$\boxed{x_0 = 10{,}000}$$

| n | x_n |
|---|---|
| 0 | 10000 |
| 1 | 10049 |
| 2 | 10097.245 |
| 3 | 10143.73123 |
| 4 | 10186.44988 |
| 5 | 10221.38213 |
| 6 | 10240.48904 |



2. Suppose that an investment of $1000 grows to $1500 in 2 years getting interest that is compounded monthly.

6 POINTS (a) What is the interest rate?

4 POINTS (b) How much interest (in dollars) does the investment earn during the first year?

(a) $x_{n+1} = \left(1+\frac{r}{12}\right)x_n \Rightarrow x_n = \left(1+\frac{r}{12}\right)^n x_0$

$x_{24} = \left(1+\frac{r}{12}\right)^{24} 1000 = 1500$

$\left(1+\frac{r}{12}\right)^{24} = 1.5$

$1 + \frac{r}{12} = 1.5^{1/24}$

$\therefore\ r = \left(1.5^{1/24} - 1\right)12 \approx .2045$ or $\boxed{20.45\%}$ *

(b) $x_{12} = \left(1+\frac{.2045}{12}\right)^{12}(1000) \approx 1224.80$

INTEREST $= x_{12} - x_0 = 1224.80 - 1000 = \boxed{\$224.80}$ *

* ANSWERS WILL VARY DEPENDING ON # OF DECIMALS USED IN CALCULATION.

3. Suppose that when a certain radioactive substance decays, its mass $M_n$ on day $n$ satisfies $M_{n+1} = aM_n$. Assuming its mass is measured to be 250 grams initially and 100 grams on day 7:

6 POINTS (a) Find the formula for the mass $M_n$.

6 POINTS (b) What is the approximate half-life of the substance, i.e., the time its takes to lose half its mass?

(a) $M_{n+1} = aM_n \Rightarrow M_n = a^n M_0$

$M_7 = a^7(250) = 100 \Rightarrow a = .4^{1/7}$

$\therefore$ $M_n = .4^{n/7}(250)$ *

* IF YOU INTERPRET 250 AS $M_1$ THEN $M_n = .4^{n-1/6}(250)$

(b) SET $M_n = \frac{1}{2}M_0$ & SOLVE FOR $n$:

$.4^{n/7}M_0 = \frac{1}{2}M_0 \Rightarrow \frac{n}{7}\ln .4 = \ln .5$

$n = \dfrac{7\ln .5}{\ln .4} \approx$ 5.3 DAYS *

* EITHER 5 OR 6 DAYS IS ALSO OK.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4. Suppose that when a certain pendulum is set in motion, it swings in such a way that the greatest positive (or negative) angle it makes on one side of the vertical is always 95% of the greatest negative (or positive) angle it previously made on the other side of the vertical, which always occurs 3 seconds earlier. If the initial angle it makes with the vertical when let go is 20°:

6 POINTS (a) How many times will the pendulum cross the vertical before the magnitude of the angle it achieves is 1° or less?

2 POINTS (b) Approximately how long will it take for this to happen?

(a) LET $x_n$ = ABSOLUTE VALUE OF THE ANGLE OF THE PENDULUM WITH VERTICAL AFTER CROSSING VERTICAL $n$ TIMES

THEN $x_{n+1} = .95\,x_n$ , $x_0 = 20$

$\Rightarrow \quad x_n = .95^n(20)$

SOLVE $x_n = .95^n(20) \leq 1$

$$.95^n \leq .05$$

$$n \ln .95 \leq \ln .05$$

$$n \geq \frac{\ln .05}{\ln .95} \qquad (\ln .95 < 0)$$

$$n \geq 58.4$$

SINCE $n$ IS AN INTEGER, $n = 59$

(b) SINCE EACH SWING OF THE PENDULUM TAKES 3 SEC,

$59 \cdot 3 = 177$ SEC

6 Points

5. Find the exact solution for $x_n$ of the homogeneous linear equation below when $x_0 = 1$.

$$x_{n+1} = \sqrt{\frac{n+2}{n+1}}x_n.$$

$$x_{n+1} = a_n x_n \quad \Rightarrow \quad x_n = \prod_{k=0}^{n-1} a_k x_0$$

$$x_{n+1} = \sqrt{\frac{n+2}{n+1}}\, x_n \quad , \quad x_0 = 1$$

$$\Rightarrow \quad x_n = \prod_{k=0}^{n-1} \sqrt{\frac{k+2}{k+1}} = \sqrt{\frac{\cancel{2}}{1}} \times \sqrt{\frac{\cancel{3}}{\cancel{2}}} \times \sqrt{\frac{\cancel{4}}{\cancel{3}}} \times \cdots \times \sqrt{\frac{n+1}{\cancel{n}}}$$

$$= \sqrt{n+1}$$

6 POINTS

6. Evaluate the geometric series

$$2 + 6 + 18 + 54 + 162 + \cdots + 2 \cdot 3^{20}.$$

IN GENERAL, $\displaystyle\sum_{k=0}^{n-1} a^k = \frac{a^n - 1}{a - 1}$

$\therefore \; 2 \displaystyle\sum_{k=0}^{20} 3^k = 2\left(\frac{3^{21} - 1}{3 - 1}\right) = \boxed{10{,}460{,}353{,}203}$

6 POINTS

7. Evaluate the infinite geometric series

$$3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \frac{3}{16} - \cdots .$$

$$3 \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k = 3 \left( \lim_{n\to\infty} \sum_{k=0}^{n-1} \left(-\frac{1}{2}\right)^k \right)$$

$$= 3 \left( \lim_{n\to\infty} \frac{1 - \left(-\frac{1}{2}\right)^k}{1 - \left(-\frac{1}{2}\right)} \right) = 3 \, \frac{1-0}{3/2} = 2$$

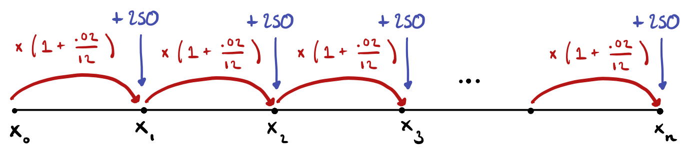
8. Every month for 20 years Amy deposits \$250 into an account that earns 2% annual interest, compounded monthly. Just after making the final deposit, Amy withdraws all of the money in the account. She then takes this large amount of money and deposits it into a second account earning 3% annual interest, compounded monthly. At the end of every month for the next 20 years, Amy makes an equal size withdrawal from the second account such that after the last withdrawal the balance in the second account is \$0.

6 Points (a) How much does Amy withdraw from the first account (and deposit into the second account)?

6 Points (b) How much does Amy withdraw from the second account each month?

4 Points (c) What is the total amount of interest that Amy earns from both accounts?

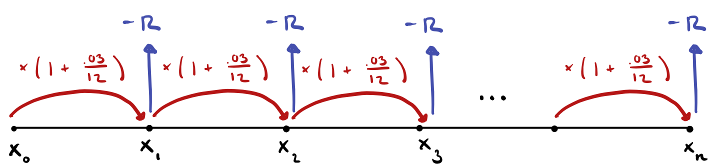
(a)




$$x_{n+1} = \left(1+\frac{.02}{12}\right)x_n + 250, \quad x_0 = 0$$

$$x_n = \left(\frac{\left(1+\frac{.02}{12}\right)^n - 1}{\left(1+\frac{.02}{12}\right) - 1}\right)250 \xrightarrow{n=12\cdot 20} x_{240} = \$73,699.21$$

(b)




$$x_{n+1} = \left(1+\frac{.03}{12}\right)x_n - R, \quad x_0 = 73,699.21 \qquad x_n = 1.0025^n(73,699.21) - R\left(\frac{1.0025^n - 1}{.0025}\right)$$

$$x_{240} = 0 \Rightarrow R = \frac{1.0025^{240}(73,699.21)}{\left(\frac{1.0025^{240}-1}{.0025}\right)} = \$408.73$$

(c) Interest = Total Deposits − Total Withdrawals

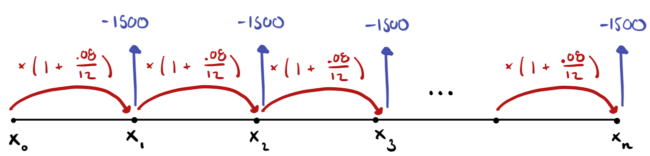
$$= 240(408.73 - 250) = \$38,095.20$$

9. Suppose a couple with a combined annual income of \$72,000 would like to purchase their first home. They have \$50,000 available as a down payment and can get a mortgage for the rest at 8% annual interest paid monthly for 30 years. However, the lender will not allow their monthly mortgage payment to exceed 1/4 of their monthly income.

5 POINTS (a) What is the maximum price home they can afford under these conditions?

5 POINTS (b) What would they have to get their annual income up to in order to afford a \$300,000 home?

(a) MAX MORTGAGE PAYMENT $R = \frac{1}{4} \times \frac{1}{12}(72{,}000) = 1500$




$$x_{n+1} = \left(1 + \frac{.08}{12}\right)x_n - 1500 \quad , \quad x_{360} = 0$$

$$x_n = \left(1 + \frac{.08}{12}\right)^n x_0 - 1500\left(\frac{\left(1+\frac{.08}{12}\right)^n - 1}{\left(1+\frac{.08}{12}\right) - 1}\right)$$

$$x_0 = \frac{1500\left(\frac{\left(1+\frac{.08}{12}\right)^{360} - 1}{\left(1+\frac{.08}{12}\right) - 1}\right)}{\left(1+\frac{.08}{12}\right)^{360}} = \$204{,}425.24$$

MAX PRICE = DOWN PAYMENT + LOAN = \$50,000 + \$204,425.24 = \$254,425.24

(b) NOW THE LOAN $x_0 = \$250{,}000$

$$0 = x_{360} = \left(1 + \frac{.08}{12}\right)^{360} 250{,}000 - R\left(\frac{\left(1+\frac{.08}{12}\right)^{360} - 1}{\left(1+\frac{.08}{12}\right) - 1}\right)$$

$$\Rightarrow \quad R = \frac{\left(1+\frac{.08}{12}\right)^{360} 250{,}000}{\left(\frac{\left(1+\frac{.08}{12}\right)^{360} - 1}{\left(1+\frac{.08}{12}\right) - 1}\right)} = \$1{,}834.41$$

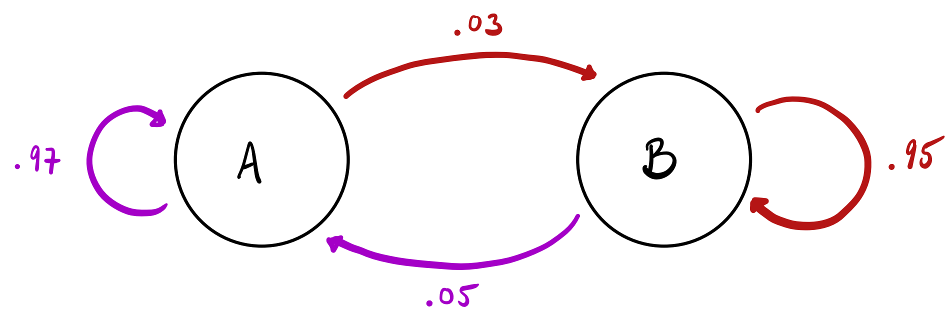
$\therefore$ ANNUAL INCOME $= 4 \times 12 \times \$1{,}834.41 = \$88{,}051.68$

10. Suppose there are two products A and B competing for market share. Each month 3% of consumers switch from using A to using B, and 5% switch from B to A. The rest stay with the one they currently use.

3 POINTS (a) Sketch the transition diagram.

5 POINTS (b) If A currently has a 70% market share and B has 30%, what percent will each have after 1 year (12 months)?

(a)




(b) $A_{n+1} = .97 A_n + .05 B_n$ $\left( B_{n+1} = .03 A_n + .95 B_n \right)$

$A_n + B_n = 1$

$\Rightarrow A_{n+1} = .97 A_n + .05 (1 - A_n)$

$A_{n+1} = .92 A_n + .05, \quad A_0 = .7$

$\therefore A_n = .92^n (.7) + \left( \frac{1-.92^n}{1-.92} \right) .05$

$A_{12} = .92^{12} (.7) + \left( \frac{1-.92^{12}}{1-.92} \right) .05 = .6526$

A WILL HAVE 65.26%, B WILL HAVE 34.74%

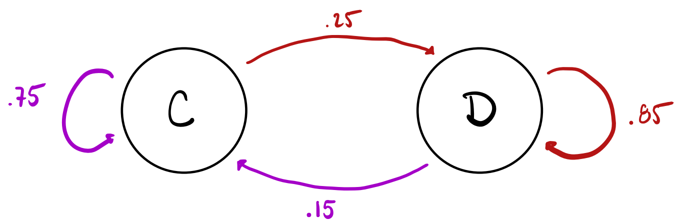
11. The alumni of Fordham University generally contribute donations or do not contribute donations according to the following pattern: 75% of those who contribute one year will contribute the next year; 15% of those who do not contribute one year will contribute the next.

3 Points (a) Sketch the transition diagram.

5 Points (b) Forty-five percent of last year's graduating class contributed this year. What percent will contribute next year? In 10 years?

4 Points (c) In the long run, what percent of alumni should the univerity expect to contribute a donation? Would this percentage change if 100% of alumni donated last year (instead of 45%)?

(a)




(b) $C_{n+1} = .75\,C_n + .15\,D_n$ $\qquad \left( D_{n+1} = .25\,C_n + .85\,D_n \right)$

$C_n + D_n = 1$

$\Rightarrow C_{n+1} = .75C_n + .15(1 - C_n) = .6C_n + .15$

$C_{n+1} = .6C_n + .15 \quad , \quad C_0 = .45$

$\Rightarrow C_n = .6^n(.45) + \left(\frac{1-.6^n}{1-.6}\right)(.15)$

$n=1: \quad C_1 = .6(.45) + .15 = .42$

$n=10: \quad C_{10} = .6^{10}(.45) + \left(\frac{1-.6^{10}}{1-.6}\right)(.15) = .3755$

$\therefore$ Next year 42%

In 10 years 37.55%

(c) $\lim_{n\to\infty} C_n = \lim_{n\to\infty} \underbrace{.6^n}_{0}(.45) + \left(\frac{1-\overbrace{.6^n}^{0}}{1-.6}\right)(.15)$

$= \frac{.15}{1-.6} = .375$

No, the value of $C_n$ converges to the stable fixed point $p = .375$, regardless of $C_0$.