# Homework 1

Sections 2.1 and 2.2

1. Consider the Annuity Saving Model

$$P_{n+1} = 1.004P_n + 500.$$

The interest is compounded monthly.

3 POINTS (a) Find the annual interest rate.

$i$ = MONTHLY INTEREST RATE

$r$ = ANNUAL INTEREST RATE

$P_n = (1+i)P_{n-1} + d$

$i = \frac{r}{12} \Rightarrow r = 12i = 12(.004)$

$r = .048$ OR $4.8\%$

3 POINTS (b) Suppose that $P_{20} = \$70,000$. Find $P_{21}$ and $P_{22}$.

$P_{21} = (1.004)(70,000) + 500 = \$70,780$

$P_{22} = (1.004)(70,780) + 500 = \$71,563.12$

4 POINTS (c) If $P_{10} = \$36,000$, what is $P_9$?

$36,000 = (1.004)P_9 + 500$

$P_9 = \frac{36,000 - 500}{1.004} \approx \$35,358.57$

2. Construct the Linear Population Model that satisfies:

4 POINTS

(a) The initial population is 3500 and the ratio of the next population to the present one is $6/5$.

$$\boxed{P_{n+1} = \frac{6}{5} P_n \ , \quad P_0 = 3500}$$

8 POINTS

(b) The initial population is 3500 and the population doubles every 3 generations.

$$P_0 = 3500$$

$$P_3 = rP_2 = r^2 P_1 = r^3 P_0 = \underbrace{r^3 P = 2 P_0}_{r = \sqrt[3]{2}} \quad \text{(DOUBLE)}$$

$$\boxed{P_{n+1} = \sqrt[3]{2}\, P_n \ , \quad P_0 = 3500}$$

3. Suppose that a contaminated body of water is being cleaned by a filtering process. Each week this process is capable of filtering out a certain fraction $a$ of all the pollutants present at that time, but another $b$ tons of pollutants seep in. Here, $a$ and $b$ are constants that satisfy $0 < a < 1$ and $b > 0$.

8 POINTS

(a) If $T_0$ tons of pollutants are initially in that body of water, write a general iterative equation for $T_n$, the amount of pollutants present $n$ weeks later.

$$\boxed{T_{n+1} = (1-a) T_n + b}$$

4 POINTS

(b) If each week 10 % of all pollutants present can be removed but another 2 tons seep in, find the values of $a$ and $b$ and write the iterative equation for this process.

$a = .1$

$b = 2$

$$\boxed{T_{n+1} = .9 T_n + 2}$$

4. Newton's *Law of Cooling* states that the rate of change of the temperature of an object is proportional to the difference of the temperature of the object and its surrounding. For instance, if $T_n$ is the temperature of a warm object in a cool room with constant temperature $R$ in $n$-th hour, then the change of the temperature $T_{n+1} - T_n$ is given by

$$T_{n+1} - T_n = k(T_n - R)$$

for some constant $k$. So we have a recursive formula

$$T_{n+1} = (1 + k)T_n - kR.$$

6 POINTS

(a) Is $k$ positive or negative? Explain your answer.

IF $T_n - R > 0$ THEN WE EXPECT THE OBJECT'S TEMPERATURE TO DECREASE : $T_{n+1} < T_n$, i.e. $T_{n+1} - T_n < 0$.

THUS $\dfrac{T_{n+1} - T_n}{T_n - R} < 0 \implies k < 0.$

10 POINTS

(b) A murder victim is discovered in an office building that is maintained at $68°F$. The average living human body temperature is $98.6°F$. Given the medical examiner found the body temperature to be $88°F$ at 8 am and that one hour later the body temperature was $86°F$, approximately at what time was the crime committed? (You don't need to find the precise time.)

$T_{n+1} = (1+k)T_n - kR$

$86 = (1+k)88 - k(68) \implies k = \dfrac{-2}{20} = -.1$

$T_n = \dfrac{T_{n+1} + kR}{1+k}$. WORKING BACKWARDS, WE HAVE :

$T_{7AM} = \dfrac{T_{8AM} + (-.1)(68)}{1 + (-.1)} = \dfrac{88 - 6.8}{.9} \approx 90.2$

$T_{6AM} = \dfrac{90.2 - 6.8}{.9} \approx 92.7$

$T_{5AM} = \dfrac{92.7 - 6.8}{.9} \approx 95.4$

$T_{4AM} = \dfrac{95.4 - 6.8}{.9} \approx 98.4$ (CLOSE TO 98.6)

~ 4AM

5. Write each in the form $x_{n+1} = a_n x_n + b_n$.

10 POINTS

(a) $x_1 = 1 - \frac{x_0}{2}, x_2 = 1 + \frac{x_1}{2}, x_3 = 1 - \frac{x_2}{2}, x_4 = 1 + \frac{x_3}{2}, \cdots$

$$x_{n+1} = \frac{(-1)^{n+1}}{2} x_n + 1$$

10 POINTS

(b) $x_1 = x_0 + 3, x_2 = \sqrt{2}x_1 + 5, x_3 = \sqrt{3}x_2 + 7, x_4 = \sqrt{4}x_3 + 9, \cdots$

$$x_{n+1} = \sqrt{n+1}\, x_n + 2n + 3$$

10 POINTS

6. Evaluate

$$\prod_{n=2}^{10000} \sqrt{\frac{n}{n-1}}.$$

$$= \sqrt{\frac{2}{1}} \cdot \sqrt{\frac{3}{2}} \cdot \ldots \cdot \sqrt{\frac{10{,}000}{9{,}999}} = \sqrt{\frac{\cancel{2}}{1} \cdot \frac{\cancel{3}}{\cancel{2}} \cdot \ldots \cdot \frac{10{,}000}{\cancel{9{,}999}}}$$

$$= \sqrt{10{,}000} = 100$$

10 POINTS

7. Find the sum

$$\sum_{n=1}^{k} n(n+1).$$

$$= \sum_{n=1}^{k} (n^2 + n) = \sum_{n=1}^{k} n^2 + \sum_{n=1}^{k} n$$

$$= \frac{k(k+1)(2k+1)}{6} + \frac{k(k+1)}{2} = \frac{k(k+1)\left[(2k+1) + 3\right]}{6}$$

$$= \frac{k(k+1)(2k+4)}{6} = \frac{k(k+1)(k+2)}{3}$$

10 POINTS

8. Evaluate

$$\sum_{n=1}^{k} n^3.$$

(Hint: Use the fact that $(n+1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1$.)

$$\begin{array}{lcccccccccc}
1^4 &=& 0^4 &+& 4\cdot 0^3 &+& 6\cdot 0^2 &+& 4\cdot 0 &+& 1 \\
2^4 &=& 1^4 &+& 4\cdot 1^3 &+& 6\cdot 1^2 &+& 4\cdot 1 &+& 1 \\
3^4 &=& 1^4 &+& 4\cdot 2^3 &+& 6\cdot 2^2 &+& 4\cdot 2 &+& 1 \\
4^4 &=& 1^4 &+& 4\cdot 3^3 &+& 6\cdot 3^2 &+& 4\cdot 3 &+& 1 \\
\vdots && \vdots && \vdots && \vdots && \vdots && \vdots \\
+\ (k+1)^4 &=& 1^4 &+& 4\cdot k^3 &+& 6\cdot k^2 &+& 4\cdot k &+& 1
\end{array}$$

$$\sum_{n=1}^{k+1} n^4 = \sum_{n=1}^{k} n^4 + 4\sum_{n=1}^{k} n^3 + 6\sum_{n=1}^{k} n^2 + 4\sum_{n=1}^{k} n + (k+1)$$

$$(k+1)^4 = 4\sum_{n=1}^{k} n^3 + 6\,\frac{k(k+1)(2k+1)}{6} + 4\,\frac{k(k+1)}{2} + (k+1)$$

$$\sum_{n=1}^{k} n^3 = \frac{1}{4}\Big[(k+1)^4 - k(k+1)(2k+1) - 2k(k+1) - (k+1)\Big]$$

$$= \frac{k+1}{4}\Big[(k+1)^3 - k(2k+3) - 1\Big]$$

$$= \frac{k+1}{4}\Big[k^3 + 3k^2 + 3k + 1 - 2k^2 - 3k - 1\Big]$$

$$= \frac{k+1}{4}\Big[k^3 + k^2\Big]$$

$$\sum_{n=1}^{k} n^3 = \frac{k^2(k+1)^2}{4} = \left(\frac{k(k+1)}{2}\right)^2$$

$$\left(\text{Note: } \sum_{n=1}^{k} n^3 = \left(\sum_{n=1}^{k} n\right)^2, \text{ Wow!!}\right)$$