Final Exam

Answer all 11 questions for a total of 100 points. Write your solutions in the accompanying blue book, and put a box around your final answers. If you solve the problems out of order, please skip pages so that your solutions stay in order. Good luck!

1. Suppose a lake is initially contaminated with 10 tons of polychlorinated biphenyls (PCBs)[1]. Each month, a cleaning process is capable of filtering out 10% of all the PCBs present, but another 1/2 ton of PCBs seeps into the lake.
   (a) (2 points) Let x_n be the number of tons of PCBs in the lake after n months. Give an iterative equation that models how the number of tons of PCBs in the lake evolves from one month to the next.
   (b) (4 points) Give an exact solution for x_n and use it do determine the amount of PCBs in the lake after one year (12 months).
   (c) (4 points) Determine the lowest PCB level that the lake will ever have under these conditions, and justify your answer.

(a) $\boxed{x_{n+1} = .9x_n + .5}$

(b) $\boxed{x_n = .9^n(10) + .5\left(\frac{1-.9^n}{1-.9}\right)} \Rightarrow \boxed{x_{12} \approx 6.4121}$

(c) SINCE COEFFICIENT OF x_n IS $.9$, A POSITIVE # WITH ABS. VAL < 1,

x_n DECREASES MONOTONICALLY TO FIXED POINT $\frac{.5}{1-.9} = \boxed{5}$

2. (a) (6 points) Evaluate $\displaystyle\sum_{n=1}^{2022}\left(5+\frac{2^n}{3^{n-1}}\right)$.
   (b) (6 points) Evaluate $\displaystyle\prod_{n=1}^{2022}\left(1+\frac{2n+1}{n^2}\right)$.

(a) $$\sum_{n=1}^{2022} 5 + \sum_{n=1}^{2022} 2\left(\frac{2}{3}\right)^{n-1}$$

$$= \boxed{2022(5) + 2\left(\frac{1-\left(\frac{2}{3}\right)^{2022}}{1-\left(\frac{2}{3}\right)}\right) \approx 10{,}110 + 6 = 10{,}116}$$

(b) $$\prod_{n=1}^{2022} \frac{n^2+2n+1}{n^2} = \prod_{n=1}^{2022} \frac{(n+1)^2}{n^2}$$

$$= \frac{\cancel{2^2}}{1^2}\cdot\frac{\cancel{3^2}}{\cancel{2^2}}\cdot\frac{\cancel{4^2}}{\cancel{3^2}}\cdot\cancel{\cdots}\cdot\frac{\cancel{2022^2}}{\cancel{2021^2}}\cdot\frac{2023^2}{\cancel{2022^2}} = \boxed{2023^2 = 4{,}092{,}529}$$

3. (6 points) Consider the following two sets of data.

n	0	1	2	3
x_n	40	10	40	70

Data set A

n	0	1	2	3
x_n	30	-50	110	-210

Data set B

One and only one of the data sets above can be modelled precisely by an autonomous linear equation. Which one? Justify your answer.

$\boxed{\text{DATA SET } B}$

NOTE THAT NO AUTONOMOUS MODEL $x_{n+1} = f(x_n)$ CAN MATCH DATA SET A :

IT WOULD REQUIRE $f(40) = 10$ AND $f(40) = 70$ $\Rightarrow\Leftarrow$.

DATA SET B FOLLOWS MODEL $x_{n+1} = -2x_n + 10$.

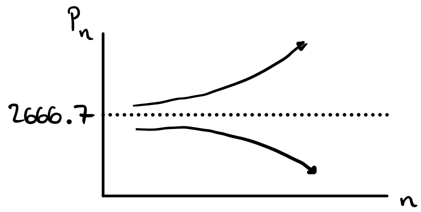
4. (a) (5 points) Consider the linear population model $P_{n+1} = 1.3P_n - 800$. For what values of P_0 does the population increase? Justify your answer.

(b) (5 points) Consider the linear population model $P_{n+1} = 0.8P_n + 500$. For what values of P_0 does the population increase? Justify your answer.

(a) COEFFICIENT 1.3 IS POSITIVE WITH ABS. VAL. > 1

$\Rightarrow$ FIXED POINT $\dfrac{-800}{1-1.3} \approx 2666.7$ IS UNSTABLE ,

AND SOLUTIONS ARE MONOTONIC.



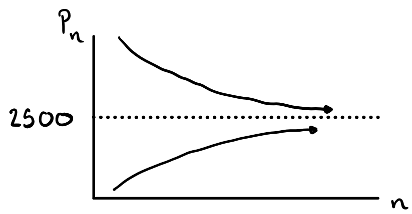
$\therefore$ POPULATION INCREASES FOR $\boxed{P_0 \geq 2667}$

$(P_0 > 2666.7)$

(b) COEFFICIENT .8 IS POSITIVE WITH ABS. VAL. < 1

$\Rightarrow$ FIXED POINT $\dfrac{500}{1-.8} = 2500$ IS STABLE ,

AND SOLUTIONS ARE MONOTONIC.



$\therefore$ POPULATION INCREASES FOR $\boxed{P_0 \leq 2499}$

$(P_0 < 2500)$

5. (4 points) Let p be the unique solution to the equation

$$x^5 + 3x^3 = \sin x + 8.$$

Assuming x_0 is sufficiently close to p, use Newton's method of root-finding to give an iterative equation $x_{n+1} = f(x_n)$ such that x_n converges to p.

SOL'N TO EQ IS ROOT OF $g(x) = x^5 + 3x^3 - \sin x - 8$

NEWTON'S METHOD: $x_{n+1} = x_n - \dfrac{g(x_n)}{g'(x_n)}$

$$\boxed{x_{n+1} = x_n - \frac{x_n^5 + 3x_n^3 - \sin x_n - 8}{5x_n^4 + 9x_n^2 - \cos x_n}}$$

6. Consider the parametrized family of functions

$$f_r(x) = rx(2 - x^2), \quad r > 0.$$

(a) (6 points) Find all fixed points and their intervals of existence.

(b) (6 points) For each of the fixed points, determine its interval of stability.

(a) FIXED POINT(S): $rx(2-x^2) = x$

$$2rx - rx^3 - x = x(2r - 1 - rx^2) = 0$$

$\boxed{\begin{array}{l} x = 0 \\ r > 0 \end{array}}$ IS FIXED POINT FOR ALL r

$x^2 = 2 - \frac{1}{r}$ $\Rightarrow$ $\boxed{\begin{array}{l} x = \pm\sqrt{2 - \frac{1}{r}} \\ r \geq \frac{1}{2} \end{array}}$ IS FIXED POINT WHEN $2 - \frac{1}{r} \geq 0$

(OR $r > \frac{1}{2}$ SINCE $\sqrt{2 - \frac{1}{r}} = 0$ WHEN $r = \frac{1}{2}$.)

(b) $f_r(x) = 2rx - rx^3$; $f_r'(x) = 2r - 3rx^2$

$x = 0$: STABLE WHEN $|f_r'(0)| < 1 \iff -1 < 2r < 1$

$\iff -\frac{1}{2} < r < \frac{1}{2}$

SINCE $r > 0$, $\boxed{\text{FIXED POINT } x = 0 \text{ HAS INTERVAL OF STABILITY } (0, \tfrac{1}{2})}$

$x = \pm\sqrt{2-\frac{1}{r}}$: STABLE WHEN $\left| f_r'\left(\pm\sqrt{2-\frac{1}{r}}\right)\right| < 1$

$$\iff \left| 2r - 3r\left(2-\frac{1}{r}\right)\right| = \left|3-4r\right| < 1$$

$$\iff \begin{array}{ccccc} -1 & < & 3-4r & < & 1 \\ -4 & < & -4r & < & -2 \\ 1 & > & r & > & \frac{1}{2} \end{array}$$

$\therefore$ $\boxed{\text{FIXED POINT } x = \pm\sqrt{2-\frac{1}{r}} \text{ HAS INTERVAL OF STABILITY } \left(\frac{1}{2}, 1\right)}$

7. (a) (5 points) Find a and b such that $\{2,4\}$ a 2-cycle of $f(x) = ax + bx^2$?
   (b) (5 points) Is the 2-cycle $\{2,4\}$ stable of unstable? Justify your answer.

(a) FIND a, b SUCH THAT $f(2) = 4$ & $f(4) = 2$

① $2a + 4b = 4$

② $4a + 16b = 2$

② - 2① : $8b = -6 \implies b = -\frac{3}{4}$

$2a + 4\left(-\frac{3}{4}\right) = 4 \implies 2a = 7 \implies a = \frac{7}{2}$

$\boxed{a = \frac{7}{2}, \quad b = \frac{-3}{4}}$

(b) $f(x) = \frac{7}{2}x - \frac{3}{4}x^2 \implies f'(x) = \frac{7}{2} - \frac{3}{2}x$

$$\left|f'(2)f'(4)\right| = \left|\left(\frac{7}{2}-3\right)\left(\frac{7}{2}-6\right)\right| = \left|\frac{1}{2}\cdot\frac{-5}{2}\right| = \frac{5}{4} > 1$$

$\therefore$ 2-CYCLE $\{2,4\}$ IS $\boxed{\text{UNSTABLE}}$

8. Let $A = \begin{bmatrix} 4 & -1 \\ 4 & -2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 12 \\ 20 \end{bmatrix}$.
   (a) (2 points) Find A^2.
   (b) (2 points) Find A^{-1}.
   (c) (4 points) Use your answer to part (b) to solve $A\vec{x} = \vec{b}$.

(a) $\begin{bmatrix} 4 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 4 & -2 \end{bmatrix} = \boxed{\begin{bmatrix} 12 & -2 \\ 8 & 0 \end{bmatrix}}$

(b) $A^{-1} = \boxed{\frac{1}{-4}\begin{bmatrix} -2 & 1 \\ -4 & 4 \end{bmatrix}} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ 1 & -1 \end{bmatrix}$

(c) $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 12 \\ 20 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ -8 \end{bmatrix}}$

9. Let

$$z = \sqrt{3} + i \quad \text{and} \quad w = 2 - i.$$

(a) (2 points) Find $z\overline{w}$.

(b) (2 points) Find $\dfrac{z}{w}$.

(c) (4 points) Find z^8. Hint: Use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

(a) $z\overline{w} = (\sqrt{3}+i)(2+i) = 2\sqrt{3} + \sqrt{3}i + 2i + i^2 = \boxed{2\sqrt{3} - 1 + (2+\sqrt{3})i}$

(b) $\dfrac{z}{w} = \dfrac{\sqrt{3}+i}{2-i} \cdot \dfrac{2+i}{2+i} = \dfrac{2\sqrt{3}-1+(2+\sqrt{3})i}{4+1} = \boxed{\dfrac{2\sqrt{3}-1}{5} + \dfrac{2+\sqrt{3}}{5} i}$

(c) $z^8 = (\sqrt{3}+i)^8 = \left(2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\right)^8$

$= \left(2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right)^8 = \left(2e^{i\frac{\pi}{6}}\right)^8$

$= 2^8 e^{i\frac{4\pi}{3}} = 256\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$= 256\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \boxed{-128 - 128\sqrt{3}i}$

10. Consider the following predator-prey-migration model.

$$P_{n+1} = P_n + 0.25Q_n - 3$$
$$Q_{n+1} = -0.5P_n + Q_n + 5$$

(a) (4 points) Find the fixed point.

(b) (4 points) Determine whether the fixed point in part (a) is stable or unstable by classifying it as a sink, source, or saddle point.

(c) (2 points) Are there any periodic points? Briefly explain your answer.

(a) $P = P + .25Q - 3 \Rightarrow .25Q = 3 \Rightarrow \boxed{Q = 12}$

$Q = -.5P + Q + 5 \Rightarrow .5P = 5 \Rightarrow \boxed{P = 10}$

(b) EIGENVALUES: $\det\begin{bmatrix} 1-\lambda & .25 \\ -.5 & 1-\lambda \end{bmatrix} = 0$

$(1-\lambda)^2 + \frac{1}{8} = 0 \Rightarrow 1-\lambda = \pm\sqrt{\frac{1}{8}}\, i$

$\lambda = 1 \pm \sqrt{\frac{1}{8}}\, i$

SINCE $|\lambda| = \sqrt{1^2 + \frac{1}{8}} > 1$, FIXED POINT IS A $\boxed{\text{SOURCE}}$

(c) $\boxed{\text{No.}}$ EACH (P_n, Q_n) LIES ON AN ELLIPSE WITH MAJOR & MINOR AXES LENGTHS SCALED BY FACTOR $|\lambda|^n$. (SOLUTIONS SPIRAL <u>OUTWARD</u> FROM THE SOURCE.)

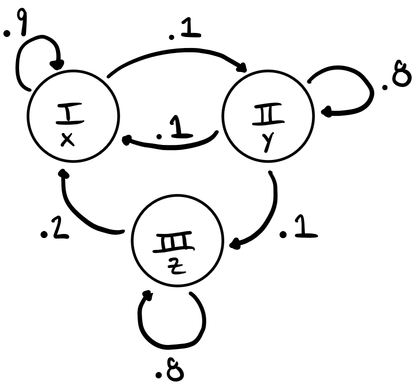
PERIODIC SOLUTIONS EXIST ONLY WHEN COMPLEX CONJUGATE EIGENVALUES BOTH HAVE ABS. VALUE / MODULUS 1.

11. Suppose a certain chronic illness has three distinct stages of severity. If someone is in Stage I, the most benign state, there is a 90% chance of remaining there, and a 10% chance of progressing to the intermediate Stage II. If someone is in Stage II there is an 80% chance of staying there, a 10% chance of going on to the most severe Stage III, but another 10% chance of returning to Stage I. If someone is in Stage III there is an 80% chance of remaining there, and a 20% chance of returning to Stage I.

(a) (2 points) Draw a transition diagram that summarizes the given information.

(b) (8 points) If the population of people with the disease is studied over an extended period of time, what proportion should we expect to see in each of the three stages of the illness at any particular time? Justify your answer.

(a)



(b)

$$x_{n+1} = .9x_n + .1y_n + .2z_n = .9x_n + .1y_n + .2(1 - x_n - y_n)$$

$$y_{n+1} = .1x_n + .8y_n$$

$$z_{n+1} = .1y_n + .8z_n$$

$$\therefore \quad x_{n+1} = .7x_n - .1y_n + .2$$

$$y_{n+1} = .1x_n + .8y_n$$

FIXED POINT:

$$x = .7x - .1y + .2 \qquad \Rightarrow \qquad .3x + .1y = .2 \quad ①$$

$$y = .1x + .8y \qquad\qquad .1x - .2y = 0 \quad ②$$

② + 2① :

$$.7x = .4 \quad \Rightarrow \quad x = 4/7$$

$$y = \tfrac{1}{2}x \quad \Rightarrow \quad y = 2/7$$

$$z = 1 - x - y = 1 - 4/7 - 2/7 = 1/7$$

$\therefore$

$$\boxed{\begin{array}{l} \text{STAGE I : } 4/7 \approx .5714 \\ \text{STAGE II : } 2/7 \approx .2857 \\ \text{STAGE III : } 1/7 \approx .1429 \end{array}}$$

$$\text{DET}\begin{bmatrix} .7-\lambda & -.1 \\ .1 & .8-\lambda \end{bmatrix} = 0 \quad \Rightarrow \quad (.7-\lambda)(.8-\lambda) + .01 = 0$$

$$\lambda^2 - 1.5\lambda + .57 = 0$$

$$\lambda = \frac{1.5 \pm \sqrt{2.25 - 2.28}}{2} = \frac{1.5 \pm \sqrt{.03}\, i}{2}$$

$$|\lambda| = \sqrt{.75^2 + \left(\frac{\sqrt{.03}}{2}\right)^2} = \sqrt{.5625 + .0075} < 1$$

$\therefore$ FIXED POINT IS SINK ✓