Exam 1

Answer all 8 questions for a total of 100 points. Write your solutions in the accompanying blue book, and put a box around your final answers. If you solve the problems out of order, please skip pages so that your solutions stay in order. Good luck!

1. (7 points) Give a linear model $x_{n+1} = a_n x_n + b_n$ such that for any initial value x_0, the sequence

$$x_0, \quad x_1 = 1 - x_0, \quad x_2 = \frac{1}{2}x_1 - 2, \quad x_3 = 4 - \frac{1}{4}x_2, \quad x_4 = \frac{1}{8}x_3 - 8, \ldots$$

is a solution.

WE HAVE

$$\begin{aligned}
x_1 &= a_0 x_0 + b_0 = -1 x_0 + 1 &&\Rightarrow a_0 = -1 && b_0 = 1\\
x_2 &= a_1 x_1 + b_1 = \tfrac{1}{2} x_1 - 2 &&\Rightarrow a_1 = \tfrac{1}{2} && b_1 = -2\\
x_3 &= a_2 x_2 + b_2 = -\tfrac{1}{4} x_2 + 4 &&\Rightarrow a_2 = -\tfrac{1}{4} && b_2 = 4\\
x_4 &= a_3 x_3 + b_3 = \tfrac{1}{8} x_3 - 8 &&\Rightarrow a_3 = \tfrac{1}{8} && b_3 = -8
\end{aligned}$$

$$\downarrow \qquad \downarrow$$

$$a_n = \frac{(-1)^{n+1}}{2^n} \qquad b_n = (-2)^n$$

$$\therefore \boxed{x_{n+1} = \frac{(-1)^{n+1}}{2^n} x_n + (-2)^n}$$

2. (7 points) Consider the linear model

$$x_{n+1} = \frac{n+3}{n+1} x_n, \qquad x_0 = 1.$$

The exact solution to this model is a product that simplifies to a relatively simple expression. Find that simplified solution for x_n.

$$\begin{aligned}
x_1 &= \tfrac{3}{1} \cdot x_0 = \tfrac{3}{1} \cdot 1\\
x_2 &= \tfrac{4}{2} \cdot x_1 = \tfrac{4}{2} \cdot \tfrac{3}{1} \cdot 1\\
x_3 &= \tfrac{5}{3} \cdot x_2 = \tfrac{5}{3} \cdot \tfrac{4}{2} \cdot \tfrac{3}{1} \cdot 1\\
x_4 &= \tfrac{6}{4} \cdot x_3 = \tfrac{6}{4} \cdot \tfrac{5}{3} \cdot \tfrac{4}{2} \cdot \tfrac{3}{1} \cdot 1
\end{aligned}$$

$$x_n = \frac{n+2}{\cancel{n}} \cdot \frac{n+1}{\cancel{n-1}} \cdot \frac{\cancel{n}}{n-2} \cdot \frac{\cancel{n-1}}{n-3} \cdot \ldots \cdot \frac{6}{\cancel{4}} \cdot \frac{5}{\cancel{3}} \cdot \frac{\cancel{4}}{2} \cdot \frac{\cancel{3}}{1} \cdot 1 = \boxed{\frac{(n+2)(n+1)}{2}}$$

3. Find the following sums *precisely*. Your answers *can* contain fractions and exponents. Your answers *cannot* be rounded decimal numbers computed with a calculator.

(a) (6 points) Find the finite sum.

$$6 - \frac{6}{7} + \frac{6}{7^2} - \frac{6}{7^3} + \frac{6}{7^4} - \frac{6}{7^5} + \ldots - \frac{6}{7^{15}}$$

(b) (6 points) Find the infinite sum (if it exists).

$$\sum_{n=0}^{\infty} \frac{3^{n+2}}{5^n}$$

(a) $\text{SUM} = 6 \sum_{k=0}^{15} \left(-\frac{1}{7}\right)^k = 6\left(\frac{1-\left(-\frac{1}{7}\right)^{16}}{1-\left(-\frac{1}{7}\right)}\right)$

$$= \frac{6\left(1-\frac{1}{7^{16}}\right)}{8/7} = \boxed{\frac{21\left(1-\frac{1}{7^{16}}\right)}{4}}$$

(b) $\sum_{n=0}^{\infty} \frac{3^2 3^n}{5^n} = 9 \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n = 9 \lim_{N \to \infty} \sum_{n=0}^{N-1} \left(\frac{3}{5}\right)^n$

$$= 9 \lim_{N \to \infty} \frac{1-\left(\frac{3}{5}\right)^n}{1-\frac{3}{5}} = \frac{9}{1-\frac{3}{5}} = \boxed{\frac{45}{2}}$$

4. In the United States, Election day is held each year on the Tuesday after the first Monday in November. Assume 70% of eligible voters who vote (V) one year will also vote the following year, and 85% of eligible voters who do not vote (D) one year will also not vote the next year.

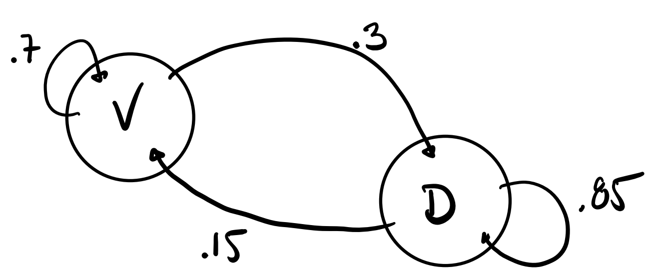
(a) (2 points) Draw a transition diagram that summarizes the survey data.

(b) (6 points) Give a linear model that describes how the proportion of eligible voters who vote V_n evolves from one year to the next.

(c) (2 points) If 50% of eligible voters voted last year, what proportion of eligible voters will vote this year?

(d) (6 points) What proportion of eligible voters should we expect to vote each year in the long run?

(a)



(b) $V_{n+1} = .7V_n + .15D_n \;;\quad V_n + D_n = 1$

$\therefore \quad V_{n+1} = .7V_n + .15(1 - V_n)$

$$\boxed{V_{n+1} = .55V_n + .15}$$

(c) $V_1 = .55V_0 + .15 = .55(.5) + .15 = \boxed{.425 \text{ OR } 42.5\%}$

(d) EXACT SOLUTION: $V_n = (.55)^n V_0 + (.15)\dfrac{1 - .55^n}{1 - .55}$

$$\lim_{n\to\infty} V_n = \lim_{n\to\infty} (.55)^n V_0 + (.15)\frac{1 - .55^n}{1 - .55}$$

with $(.55)^n \to 0$ and $.55^n \to 0$

$$= \frac{.15}{1 - .55} = \frac{.15}{.45} = \boxed{\frac{1}{3}}$$

5. Uh oh! You've discovered a colony of P_0 ants living in your kitchen. Experts tell you that if you do nothing, the colony will increase its population by 20% every week. So you immediately borrow your friend's pet lizard that eats exactly 400 ants every week.

(a) (8 points) Give a linear model that describes how the ant population in your kitchen after n weeks P_n evolves from one week to the next.

(b) (2 points) Is your model from part (a) autonomous? Is it homogeneous?

(c) (6 points) For what value(s) of P_0 (if any) will the lizard be able to eradicate the ant colony? For what values of P_0 (if any) will the lizard fail to eradicate the ant colony? Justify your answers.

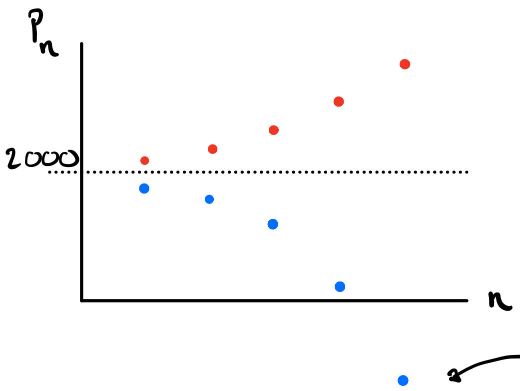
(a) $\boxed{P_{n+1} = 1.2P_n - 400}$

(b) $\boxed{\text{AUTONOMOUS, NOT HOMOGENEOUS}}$

(c) NOTE THAT FIXED POINT $p = \frac{b}{1-a} = \frac{-400}{1-1.2} = 2000$

IS UNSTABLE SINCE $|a| = |1.2| > 1$.

SINCE $a = 1.2 > 0$, SOLUTIONS DO NOT OSCILLATE



$\therefore$ SOLUTIONS WITH $P_0 < 2000$

WILL DIVERGE TO $-\infty$.

(THE MODEL WILL GIVE NEGATIVE VALUES, BUT THERE WILL BE NO ANTS)

THUS, THE LIZARD WILL ERADICATE THE COLONY IF $P_0 < 2000$.
THE LIZARD WILL FAIL TO ERADICATE THE COLONY IF $P_0 \geq 2000$.

6. Let

$$f(x, y) = x^3 + y^3 - 3x - 12y + 5.$$

(a) (6 points) Find the partial derivatives f_x and f_y.

(b) (6 points) Find the critical points of f. That is, find all points (a, b) such that

$$f_x(a, b) = f_y(a, b) = 0.$$

(a) $f_x = 3x^2 - 3$
$f_y = 3y^2 - 12$

(b) $f_x = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$
$f_y = 3y^2 - 12 = 0 \quad y = \pm 2$

CRITICAL POINTS: $\boxed{(\pm 1, \pm 2)}$ OR $(1,2), (1,-2), (-1,2), (-1,-2)$

7. Suppose $p = 100$ is a fixed point of the linear model

$$x_{n+1} = ax_n + 80.$$

(a) (6 points) Find a.

(b) (2 points) Is the fixed point p stable of unstable?

(c) (4 points) For what values of x_0 would solutions monotonically decrease?

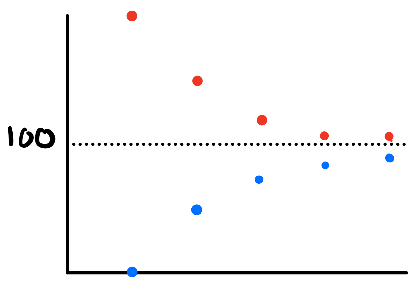
(d) (4 points) For what values of x_0 would solutions oscillate above/below the fixed point p?

(a) FIXED POINT EQUATION: $p = f(p) \Rightarrow p = ap + 80$

$p = 100 \Rightarrow 100 = 100a + 80 \Rightarrow \boxed{a = \frac{1}{5}}$

(b) SINCE $|a| = |\frac{1}{5}| < 1$, p IS $\boxed{\text{STABLE}}$

(c)



SINCE $a > 0$, OSCILLATION DOES NOT OCCUR. SOLUTIONS MONOTONICALLY DECREASE WHEN $x_0 \geq p$, i.e.

$$\boxed{x_0 \geq 100}$$

(d) $\boxed{\text{NONE}}$

8. Consider the following two sets of data.

n	0	1	2	3
x_n	10	70	-110	430

(a) Data set A

n	0	1	2	3
x_n	10	30	150	450

(b) Data set B

One and only one of the data sets above can be modelled precisely by an autonomous linear equation.

(a) (6 points) Which one? Justify your answer geometrically.

(b) (8 points) Find the autonomous linear model that fits that data set.

(a) $x_{n+1} = f(x_n)$

$\Rightarrow (x_n, x_{n+1})$ IS ON THE GRAPH OF $f(x)$

A: $\underbrace{(10,70),\ (70,-110)}_{(1)}$, $(-110, 430)$, with $\underbrace{(70,-110),\ (-110,430)}_{(2)}$

$$\text{SLOPE CONNECTING } (1) = \frac{-110-70}{70-10} = \frac{-180}{60} = -3$$

$$\text{SLOPE CONNECTING } (2) = \frac{430-(-110)}{-110-70} = \frac{540}{-180} = -3$$

SAME! ✓

∴ THE 3 POINTS ARE COLINEAR

$\boxed{\text{DATA SET A FOLLOWS A LINEAR MODEL PRECISELY.}}$

B: $\underbrace{(10,30),\ (30,150)}_{(1)}$, $(150, 450)$, with $\underbrace{(30,150),\ (150,450)}_{(2)}$

$$\text{SLOPE CONNECTING } (1) = \frac{150-30}{30-10} = 6$$

$$\text{SLOPE CONNECTING } (2) = \frac{450-150}{150-30} = \frac{5}{2}$$

NOT THE SAME! ✗

∴ THE THREE POINTS ARE NOT COLINEAR.

DATA SET B DOES NOT FOLLOW A LINEAR MODEL PRECISELY.

(b)

$$\begin{array}{r} 70 = 10a + b \\ -\ (-110 = 70a + b\ ) \\ \hline 180 = -60a \end{array}$$

$\Rightarrow a = -3,\ b = 100$

$$\boxed{x_{n+1} = -3x_n + 100}$$